Elasticity, Shape Fluctuations and Phase Transitions in the New Tubule Phase of Anisotropic Tethered Membranes

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We study the shape, elasticity and fluctuations of the recently predicted[1] and subsequently observed (in numerical simulations)[2] tubule phase of anisotropic membranes, as well as the phase transitions into and out of it. This novel phase lies between the previously predicted flat and crumpled phases, both in temperature and in its physical properties: it is crumpled in one direction, and extended in the other. Its shape and elastic properties are characterized by a radius of gyration exponent ν and an anisotropy exponent z. We derive scaling laws for the radius of gyration $R_G(L_\perp, L_y)$ (i.e. the average thickness) of the tubule about a spontaneously selected straight axis and for the tubule undulations $h_{rms}(L_\perp, L_y)$ transverse to its average extension. We show that for square membranes (with *intrinsic* size $L_\perp = L_y = L$), $R_G \propto L^\nu$, and $h_{rms} \propto L^{1-\eta_\kappa z/2}$, with η_κ a bending rigidity anomalous elasticity exponent related to ν and z. For phantom (i.e. *non*-self-avoiding) membranes, we predict $\nu = 1/4$, $z = 1/2$ and $\eta_\kappa = 0$, *exactly*, in excellent agreement with simulations. For $D = 2$ dimensional membranes embedded in the space of dimension $d < 11$, self-avoidance greatly swells the tubule and suppresses its wild transverse undulations, changing its shape exponents ν, z, and η_κ. For a D-dimensional membrane embedded in $d > d_*$ ($d_*(D = 2) > 7/2$), $\eta_\kappa = 0$ and $z = (D-1+2\nu)/3$, while for $d < d_*$, $\eta_\kappa > 0$ and $z = (D-1+2\nu)/(3-\eta_\kappa)$. "Flory" theory yields, in the physical case of $D = 2$ and $d = 3$, $\nu = 3/4$, while the recent $11 - \epsilon$ expansion results[3] yields $\nu = 0.52$. The actual value of ν probably lies closer to the Flory estimate, between these two limits. We give detailed scaling results for the shape of the tubule of an arbitrary aspect ratio, i.e. for the tubule thickness, its transverse undulations, and a variety of other correlation functions, as well as for the anomalous elasticity of the tubules, in terms of ν and z. Finally we present a scaling theory for the shape and specific heat near the continuous transitions into and out of the tubule phase and perform detailed renormalization group calculations for the crumpled-to-tubule transition for phantom membranes.

I. INTRODUCTION

Tethered membranes[4] became a subject of great interest when it was theoretically predicted[5] that, unlike polymers, which are always orientationally disordered, membranes can exhibit two distinct phases: crumpled and flat, with a "crumpling" transition between them. The flat phase is particularly novel and intriguing, because it provides an example of a two dimensional system with a continuous symmetry that nonetheless exhibits a long-ranged order (specifically, long-ranged orientational order in the normal to the membrane) in apparent violation of the Hohenberg-Mermin-Wagner theorem[6]. This ordering is made possible by "anomalous elasticity"[5,7,8]: thermal fluctuations infinitely enhance the bending rigidity κ of the membrane at long wavelengths, thereby stabilizing the orientational order against these very fluctuations. This is perhaps the most dramatic illustration yet found of the phenomenon of "order from disorder".

Rich as these phenomena are, most past theoretical work[4] has been restricted to *isotropic* membranes. In a recent paper[1] we extended these considerations to *intrinsically anisotropic* membranes (e.g., polymerized membranes with in-plane tilt order[9]) and found, astonishingly, that anisotropy, a seemingly innocuous generalization, actually leads to a wealth of new phenomena. Most dramatically, we found an entire new phase of membranes, which we called the "tubule" phase, ubiquitously intervenes between the high temperature crumpled and low temperature "flat" phases. The defining property of the tubule phase is that it is crumpled in one of the two membrane directions, but "flat" (i.e., extended) in the other. Its average shape is a long, thin cylinder of length $R_y = L_y \times O(1)$ and radius $R_G(L_\perp) \ll L_\perp$, where L_y and $L_\perp$ are the dimensions the membrane would have in the extended and crumpled directions respectively, were it to be flattened out. It should be clarified here that we use the term "cylinder" *extremely* loosely; as illustrated in Fig.2, a cross section of the membrane perpendicular to the tubule axis (y) will look as disordered as a flexible

polymer. These tubules, occurring as a low temperature phase of anisotropic *polymerized* membranes, have little in common (and therefore should not be confused) with micro-tubules that are found in *liquid* phospholipid membranes[10].

Only in the special case of perfectly isotropic membranes[11] is it possible for the membrane to undergo a direct transition from the flat to the crumpled phase. The theoretically predicted[1] and recently observed[2] phase diagram is shown in Fig.1.
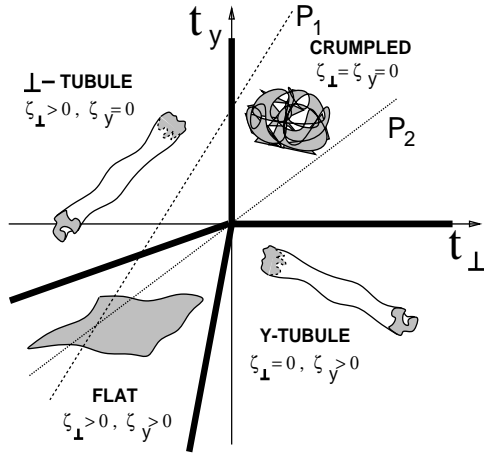

FIG. 1. Phase diagram for anisotropic tethered membranes showing the new tubule and previously studied flat and crumpled phases.

The direct crumpling transition studied previously occurs in our more generic model only for that special set of cuts through the phase diagram (like P_2) that pass through the origin. Generic paths (like P_1) will experience *two* phase transitions, crumpled-to-tubule, and tubule-to-flat, that are in new, heretofore uninvestigated universality classes.

This prediction was recently dramatically confirmed in Monte Carlo simulations of phantom (i.e., non-self-avoiding) membranes by Bowick, Falcioni and Thorleifsson (BFT).[2]. They simulated membranes with different bare bending moduli κ_x and κ_y in the orthogonal x and y directions. As temperature (or one of the bending rigidities e.g. κ_x) is varied, we predicted our model would follow a generic path like P_1 in Fig.1. And, indeed, these simulations[2] observed two specific heat bumps, corresponding to two distinct continuous transitions crumpled-to-tubule and tubule-to-flat (rounded by finite membrane size), just as we predicted[1]. Furthermore, the shape of the membrane in the phase between these two transitions was exactly that of the tubule above (see Figure 2), and had, within numerical errors, precisely the scaling properties and exponents that we predicted for phantom tubules[1]. Here we present our detailed study of these transitions and the tubule phase, in the presence of both thermal fluctuations and self-avoidance.

There are a number of possible experimental realizations of anisotropic membranes. One is polymerized

membranes with in-plane tilt order[9]. *Fluid* membranes with such order have already been found[12,13]; it should be possible to polymerize these without destroying the tilt order. Secondly, membranes could be fabricated by cross-linking DNA molecules trapped in a fluid membrane[12,13]. Performing the cross-linking in an applied electric field would align the DNA and "freeze in" the anisotropy induced by the electric field, which could then be removed.

The tubule cross-sectional radius R_G, (hereafter called the radius of gyration), and its undulations h_{rms} transverse to its average axis of orientation, obey the scaling laws:

$$R_G(L_\perp, L_y) = L_\perp^\nu S_R(L_y/L_\perp^z) , \qquad (1.1)$$

$$h_{rms}(L_\perp, L_y) = L_y^\zeta S_h(L_y/L_\perp^z) , \qquad (1.2)$$

where $\zeta = \nu/z$,

$$z = \frac{1}{3 - \eta_\kappa}(1 + 2\nu) , \qquad (1.3)$$

we have specialized in Eq.1.3 to $D = 2$ (with general expression for a D-dimensional membrane given in the main text), the universal exponents ν and z are < 1, η_κ is the anomalous elasticity exponent for the tubule bending rigidity κ (as defined by $\kappa \sim L_y^{\eta_\kappa}$, also see below), and for convenience we chose to measure the intrinsic lengths L_y and $L_\perp$ in units of the ultraviolet cutoff, set approximately by the the monomer (e.g. phospholipid) size.
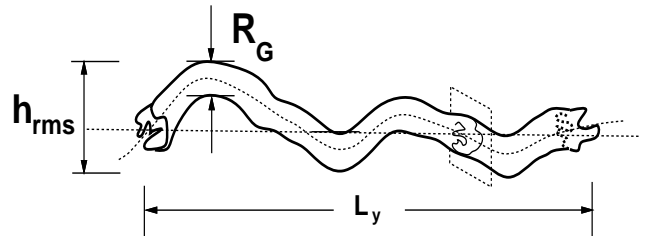

FIG. 2. Schematic picture of the tubule phase of anisotropic polymerized membrane, with the definition of its thickness R_G and roughness h_{rms}, our predictions for which are given in Eqs.1.1 and 1.2.

The scaling functions $S_{R,h}(x)$ have the limiting forms:

$$S_R(x) \propto \begin{cases} x^{\zeta - \nu_p/z}, & \text{for } x \to 0 \\ \text{constant}, & \text{for } x \to \infty \end{cases} \qquad (1.4)$$

$$S_h(x) \propto \begin{cases} \text{constant}, & \text{for } x \to 0 \\ x^{\frac{3}{2} - \zeta}, & \text{for } x \to \infty \end{cases} , \qquad (1.5)$$

where ν_p is the radius of gyration exponent of a coiled linear polymer $\approx 3/5$. These scaling functions are *universal* (i.e., independent of material parameters and temperature), up to an overall non-universal multiplicative factor, which can, and will, depend on material parameters and temperature.

The scaling forms, Eq.1.4 and 1.5 imply that for a "roughly square" membrane – that is, one with $L_\perp \sim L_y \equiv L$ – in the limit $L \to \infty$

$$R_G(L_\perp \sim L_y \equiv L) \propto L^\nu , \tag{1.6}$$

$$h_{rms}(L_\perp \sim L_y \equiv L) \propto L^{1-\eta_\kappa z/2} , \tag{1.7}$$

where we have used the fact that for $L_y \sim L_\perp$, the argument $x \equiv L_y/L_\perp^z$ of the scaling functions $S_{R,h}(x)$ goes to infinity as $L \to \infty$, and used Eq.1.3 to simplify Eq.1.7.

Detailed renormalization group calculations show that η_κ is strictly positive. Hence, $h_{rms} << L$ for a roughly square membrane as $L \to \infty$. Thus, the end-to-end orientational fluctuations $\theta \sim h_{rms}/L \propto L^{-\eta_\kappa z/2} \to 0$ as $L \to \infty$ for such a roughly square membrane, proving that tubule order (which requires orientational persistence in the extended direction) *is* stable against undulations of the tubule embedded in $d = 3$ dimensions.

On the other hand, in the limit $L_y >> L_\perp$, in which the tubule looks more and more like a linear polymer (a ribbon of width $L_\perp$ and length L_y), we find

$$h_{rms} \propto \frac{L_y^{3/2}}{L_\perp^{z(3/2-\zeta)}} = \frac{L_y^{3/2}}{L_\perp^{1/2+\eta_\kappa z/2}} \equiv L_y \left(\frac{L_y}{L_P(L_\perp)} \right)^{1/2} , \tag{1.8}$$

acting like a rigid polymer with a *polymer* bending rigidity

$$\kappa_p(L_\perp) \propto L_\perp^{1+z\eta_\kappa} . \tag{1.9}$$

It is well known[14], of course, that a linear polymer does *not* have long-ranged orientational order i.e., it has a finite orientational persistence length L_P. For length smaller than $L_P(L_\perp)$ we recover the well-known[14] $L_y^{3/2}$ growth of transverse fluctuations. By equating h_{rms} from Eq.1.8 with the length L_y of the tubule, and defining (ribbon width-dependent persistent length) $L_P(L_\perp)$ to be the value of L_y at which this equality occurs, we obtain an estimate for the orientational persistence length L_P of a long, skinny tubule:

$$L_P(L_\perp) \propto L_\perp^{1+\eta_\kappa z} . \tag{1.10}$$

We see that only *very* long, skinny membranes ($L_y >> L_\perp$) will be orientationally disordered; for any membrane with a reasonable aspect ratio (i.e., $L_y \sim L_\perp$), L_y is much less than $L_P(L_\perp)$, and the orientational order of the tubule persists throughout it. This proves that the tubule phase is stable in the thermodynamic limit against thermal fluctuations.

Equation 1.9 indicates that the effective polymer bend modulus $\kappa_p(L_\perp)$ is "anomalous", by which we mean the fact that $\kappa_p(L_\perp)$, grows as a power of $L_\perp$ greater (by the "anomalous dimension" $\eta_\kappa z$) than 1 (naively expected based on dimensional analysis). This together with the concomitant anomalous dimension of the persistent length $L_P(L_\perp)$, Eq.1.10 embodies the phenomenon

known as "anomalous elasticity".[15,5,7,8] In addition to fluctuating membranes, they have consequences for polymers whose internal structure is that of a long ribbon of dimension $L_\perp \times L_y$, with $L_y >> L_P(L_\perp) >> L_\perp$. Provided that $L_\perp$ is large enough that the anomalous elasticity can manifest itself, the radius of gyration $\mathcal{R}_G^p$ of this polymer (which, since $L_y >> L_P$, will be coiled) will, in fact, grow more rapidly with the *transverse* dimension $L_\perp$ of the polymer than the conventional elastic theory would predict. Specifically, we expect:

$$\mathcal{R}_G^p \approx L_P(L_\perp) \left(\frac{L_y}{L_P(L_\perp)} \right)^{\nu_p} ,$$

$$\propto L_y^{\nu_p} L_\perp^{(1-\nu_p)(1+\eta_\kappa z)} , \tag{1.11}$$

while conventional elastic theory would imply $\mathcal{R}_G^p \propto L_\perp^{1-\nu_p}$.

In addition to this anomalous elasticity in the effective polymer bend modulus, the fluctuating tubule also displays anomalous elasticity for stretching the tubule. In particular, experiments that attempt to measure the stretching modulus g_y of the tubule (defined more precisely by the renormalized version of Eqs.5.5 and 5.7) at wavevector $\mathbf{q}$ will produce results that depend strongly on $\mathbf{q}$, *even in the limit* $\mathbf{q} \to 0$. In particular, this apparent wavevector-dependent stretching modulus $g_y(\mathbf{q})$ *vanishes* as $|\mathbf{q}| \to 0$, according to the scaling law

$$g_y(\mathbf{q}) = q_y^{\eta_u} S_g(q_y/q_\perp^z) , \tag{1.12}$$

where $\eta_u > 0$ is another universal exponent, and $S_g(x)$ another universal scaling function.

Similarly, the tubule bend modulus κ (also defined more precisely by the renormalized version of Eqs.5.4 and 5.6) becomes strongly wavevector dependent as $\mathbf{q} \to 0$, but it *diverges* in that limit:

$$\kappa(\mathbf{q}) = q_y^{-\eta_\kappa} S_\kappa(q_y/q_\perp^z) , \tag{1.13}$$

with $\eta_\kappa \geq 0$ yet another universal exponent, and $S_\kappa(x)$ yet another universal scaling function.

The relations Eqs.1.1-1.2 summarize all of the scaling properties in terms of the two universal exponents ν and z (or equivalently η_κ). Clearly, we would like to predict their numerical values. There are three distinct cases to be considered, as we decrease the embedding dimension d in which the $D = 2$-dimensional membrane fluctuates, as illustrated in Fig.3 (the generalization to arbitrary D is given in the main text).
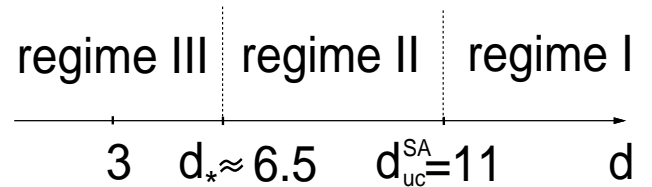
FIG. 3. Illustration (in $D = 2$) of the three regimes of embedding dimension d with qualitatively and quantitatively different tubule shape scaling properties. Our estimates of $d_* \approx 6.5$ place the physical tubule ($d = 3$) deep in regime III; the strict bound $d_* > 7/2$ *guarantees* this.

Regime I:
For a phantom membrane, or for a membrane with intrinsic dimension $D = 2$ embedded in a space of dimension $d \geq d_{uc} = 11$, self-avoidance effects can be asymptotically ignored in the tubule phase, and we predict[1]

$$\nu = \frac{1}{4} , \tag{1.14}$$

$$z = \frac{1}{2} , \tag{1.15}$$

$$\eta_\kappa = 0 , \tag{1.16}$$

$$\eta_u = 1 , \tag{1.17}$$

Regime II:
For a self-avoiding membrane with $d_* < d < d_{uc} = 11$ (with $d_* > 7/2$), we have shown (as we describe in detail in Sec.VI) that the bending elasticity is *not* anomalous, i.e., $\eta_\kappa = 0$, as guaranteed by an *exact* "tubule-gauge" symmetry (see Sec.VI B). This, using Eq.1.3 immediately leads to the the exponent relation, $z = (1 + 2\nu)/3$, which states that for $d > d_*$ all properties of a self-avoiding tubule can be expressed in terms of a single radius of gyration exponent ν. In this range $d_* < d < d_{uc} = 11$ of embedding dimensionality, the exponents ν and z can be computed in an $\epsilon = 11 - d$-expansion. This has been done recently by Bowick and Guitter (BG)[3] who have verified the validity of the Ward identity $z = (1 + 2\nu)/3$ (for $D = 2$) perturbatively, to all orders in ϵ. Furthermore, for *all* embedding dimensions $d > d_*$, the absence of anomalous bend elasticity (i.e., $\eta_\kappa = 0$) renders the self-avoiding interaction ineffective in stabilizing wild transverse tubule undulations and for a square membranes, Eqs.1.7 and 1.10 show that the $D = 2$-dimensional tubule phase is only *marginally* stable. For $D = 2$, this $d_* < d < d_{uc} = 11$ regime has:

$$\frac{2}{5} > \nu > \frac{1}{4} , \tag{1.18}$$

$$z = \frac{1}{3}(1 + 2\nu) , \tag{1.19}$$

$$\eta_\kappa = 0 , \tag{1.20}$$

$$\eta_u = 3 - \frac{1}{z} , \tag{1.21}$$

Regime III:
Finally, as we describe in Sec.VI, the physics of the *physical* tubule (i.e., $D = 2$-dimensional tubule embedded in $d = 3$ dimensions) is much richer than that for the embedding dimensions $d > d_*$, where "tubule-gauge" symmetry imposes strict *non*renormalization of the tubule bending rigidity κ. For $d < d_*$, because of the presence of additional elastic nonlinearities (which are irrelevant for d near $d_{uc} = 11$, but become strongly relevant for

physical dimensionality $d < d_*$), this ϵ-expansion about $d = d_{uc} = 11$ gives no information about the simultaneous role that the self-avoidance and elastic nonlinearity play in the physical tubule ($D = 2$, $d = 3 < d_*(D = 2)$), where they are *both* important. We find that, as the embedding dimension d is lowered below $d_* < d_{uc} = 11$ ($d_*(D = 2) > 7/2$), the nonlinear elasticity becomes relevant, destabilizing the fixed point studied in Ref. 3, and leading to the breakdown of the $z = (1 + 2\nu)/3$ relation (with the amount of breakdown described by a new anomalous elasticity exponent η_κ). Hence physical tubules ($D = 2$, $d = 3$) are described by a new infra-red stable fixed point, that is non-perturbative in $\epsilon = 11 - d$, which incorporates the simultaneous effects of self-avoidance and nonlinear anomalous elasticity. This new fixed point characterizes the $d < d_*$ regime (appropriate to a physical tubule) with shape scaling exponents

$$\nu \geq \frac{2}{5} , \tag{1.22}$$

$$\nu > \frac{1}{d - 1} , \tag{1.23}$$

$$z = \frac{1}{3 - \eta_\kappa}(1 + 2\nu) , \tag{1.24}$$

$$2\eta_\kappa + \eta_u = 3 - \frac{1}{z} , \tag{1.25}$$

$$\eta_\kappa , \eta_u > 0 , \tag{1.26}$$

We cannot calculate exactly the critical embedding dimension $d_*(D)$ that separates regime II and regime III, but we *can* derive a *rigorous* lower bound on it $d_*(2) > 7/2$. Thus the physical tubule, $D = 2$, $d = 3$ falls in regime III. Our best estimate of $d_*(2)$ is that it lies between 5 and 7.

It should be emphasized that all of the exponents are *universal* in a given embedding dimension d. Indeed, for $d_* < d < 11$, where *all* of the exponents are determined by the single unknown exponent ν, there are two different analytical approximations to ν that agree to better than 1% for $d > 8$, and to better than 10% for d's greater than the likely values of d_*. These analytical methods are: Flory theory[1], which predicts

$$\nu_F = \frac{3}{d + 1} , \tag{1.27}$$

and the leading order in $\epsilon = 11 - d$ expansion of Bowick and Guitter[3], which gives,

$$\nu_\epsilon = \frac{3}{4 - c\epsilon} - \frac{1}{2} , \tag{1.28}$$

with

$$c = 0.13125 , \tag{1.29}$$

We suspect, based on the experience of comparing polymer exponents obtained from Flory theory with those

obtained from the ϵ-expansion, that, although BG's results are certainly more accurate near $d = 11$, when the BG and Flory results start to disagree appreciably (i.e., below $d = 7$), the Flory result is probably the more accurate. Nonetheless, the *extremely* close agreement between these two very different approaches in these high embedding dimensions increases our faith in both of them.

In fact, as we describe in detail in Sec.VI B, for $D = 2$-dimensional membrane, d_* is determined by the condition that $\nu(d) \to 2/5$ as $d \to d_*^+$. Using the Flory result (Eq.1.27), this gives $d_* = 13/2 = 6.5$; while using the BG result (Eq.1.28) gives $d_* = 11 - 2/(3c) = 5.92$.

All of the exponents jump discontinuously (as a function of d) at d_*; figure 9 shows such a plot, schematically, for $\nu(d)$ and $\eta_\kappa(d)$.

For a physical tubule, Flory theory, Eq.1.27, implies

$$\nu_F(D = 2, d = 3) = 3/4 , \qquad (1.30)$$

in contrast to the BG result Eq.1.28, which implies $\nu_\epsilon(D = 2, d = 3) = 0.517$. What is the correct value of ν in $d = 3$? As discussed above, our experience with polymers suggests that Flory theory is more reliable[16] than the ϵ-expansion when both are pushed well below the upper critical dimension. One might be concerned that this ceases to be true for tubules, due to the discontinuous behavior of all of the exponents at d_*, but we will present arguments later that suggest that this is not the case, and that Flory theory is probably quite accurate in the physical case of $d = 3$.

It is widely[17,18], though not universally,[19,20] believed that self-avoidance destroys the crumpled phase. What is definitely known is that the crumpled phase has only been seen in simulations of phantom membranes and in more recent simulations by Baumgartner[19] of a self-avoiding plaquette membrane model. It is therefore reasonable to ask whether our tubule phase will suffer the same fate. We think not, for the following reasons:

1. It is clear that self-avoidance, though a relevant perturbation (in physical embedding dimension $d < d_{uc} = 11$) has far less effect on the tubule than the crumpled phase, since points on the membrane widely separated in the y-direction never bump into each other in the tubule phase, but do in the crumpled phase.

2. The *analytic* argument that self-avoidance destroys the crumpled phase is based on the Gaussian variational (GV) approximation[21,22], which predicts that the radius of gyration exponent $\nu_{GV}^{crumpled} = 4/d$, which implies that $\nu \geq 1$ for $d \leq 4$, and hence that the membrane is extended (i.e. flat) for those dimensions (which, of course, include the physical case of $d = 3$). We find that the same Gaussian variational approximation leads to the same conclusion for the tubule phase. Our result for $D = 2$ is

$$\nu_{GV}^{tubule} = \frac{7}{3d - 5} , \qquad (1.31)$$

and implies $\nu_{GV}^{tubule} \geq 1$ for $d \leq 4$, and hence an instability of the self-avoiding tubule to an extended (i.e. flat) membrane in physical dimensions.

We are not, however, overly concerned by this result, for a number of reasons:

(a) The Gaussian variational approximation is known to be far from trustworthy. For example, it predicts $\nu = 2/d$ for linear polymers, which not only is less accurate for *all* d between 1 and 4 than the Flory result $\nu = 3/(d + 2)$, but also incorrectly predicts that the lower critical dimension d_{lc} below which linear polymers are always extended is $d_{lc} = 2$, whereas, in fact, it is known exactly that $d_{lc} = 1$, a result that is also predicted exactly by the Flory theory. Thus, the Gaussian variational approximation is *very* unreliable in predicting the lower critical dimension of a crumpled object.

(b) There is a good reason to believe it is equally unreliable for our problem as well. If we compare the Flory prediction for ν with the ϵ-expansion calculation of Ref. 3 (which is asymptotically exact in $d \to 11$), in, e.g., $d = 8$, we find they differ less than $1/3$ of 1%: $\nu_\epsilon = 0.332$ [3], $\nu_{Flory} = 1/3$ [1]; while the Gaussian variational result $\nu_{GV}^{tubule} = 7/19 = 0.3684$ is nearly 40 times as far off ν_ϵ as the Flory result. This strongly suggests that both Flory theory and the ϵ-expansion are more reliable than the Gaussian variational approximation, and both of them predict ν substantially < 1 in $d = 3$: $\nu_F = 3/4$ [1], $\nu_\epsilon = 0.517$ [3].

(c) Finally, on more general grounds, while the Gaussian variational method can be quite useful, only some of its results can be trusted. Certainly it is likely that the *trends* of, e.g., exponents with dimensionality d and D, are captured correctly by this theory. The very existence of the crumpled phase relies on the precise value of $\nu(d)$ (it disappears if the $d < d_{lc}$, with d_{lc} defined by $\nu(d_{lc}) = 1$). However, as with any approximate method, especially with uncontrolled approximations such as the Gaussian method, there is little credibility in the actual *values* of the exponents. Furthermore, the Gaussian variational approximation is very closely related to a large expansion in $1/d$ about the embedding dimension $d \to \infty$ limit.[23] It is therefore intrinsically untrustworthy and ad hoc for small values of d at which one is assessing the stability of the tubule (or crumpled) phase, which very delicately and

sensitively depends on the precise value of ν at small d.

In the remainder of this paper we present the details of our calculations. In Sec.II we introduce the Landau-Ginzburg-Wilson free energy for our generalized model of anisotropic polymerized membranes. In Sec.III we will first solve this model in mean field theory. From this solution we obtain the phase diagram for anisotropic polymerized membranes, and identify and characterize the new tubule phase as well as the previously studied crumpled and flat phases. In Sec.IV we show that the scaling properties of the flat and crumpled phases are unaffected by the anisotropy. In Secs.V and VI we then consider the effects of both thermal fluctuations and self-avoidance on the tubule phase. We treat this problem using Flory theory, renormalization group and Gaussian variational methods. We calculate the upper critical embedding and intrinsic dimensions for both effects, and thereby show that both are relevant for the physical case of two-dimensional membranes embedded in three dimensions. We also show that, although there is *no* anomalous elasticity for the bend modulus κ along the tubule near $d = d_{uc} = 11$ (due to aforementioned "tubule gauge" symmetry), such anomaly must set in for embedding dimensions $d < d_*$, with $d_* > 7/2$. When this happens, the fixed point (perturbative around $d = 11$)[3] which describes a self-avoiding (*bend* elastically *non*-anomalous) tubule, becomes unstable, and a new fixed point controls the tubule phase. We derive new *exact* relations Eqs.6.62 and 6.63 between ν and z, which involve anomalous elasticity exponent η_κ (or η_u, related to it) and are appropriate for a physical (with anharmonic elasticity) tubule, described by this new fixed point. We then use the Flory[1] and extrapolated $\epsilon = 11 - d$–expansion[3] results for ν in this relation to determine z and all other tubule shape exponents in terms of two constants that, unfortunately, we were not able to compute accurately.

In Section V we also derive the scaling results Eq.1.1 and Eq.1.2 for R_G and h_{rms}, and for the anomalous elastic theory as well.

In Section VII we use the renormalization group to analyze the crumpled-to-tubule transition. We then construct a scaling theory of the crumpled-to-tubule and tubule-to-flat transitions, and compute within Flory theory the critical exponents for these transitions.

In Section VIII we summarize, conclude, and make some suggestions for further analytic, numerical, and experimental work.

II. MODEL

Our model for anisotropic membranes is a generalization of the isotropic model considered in Ref. 24. As there, we characterize the configuration of the membrane by giving the position $\vec{r}(\mathbf{x})$ in the d-dimensional embedding space, of the point in the membrane labeled by a D-

dimensional internal co-ordinate $\mathbf{x}$. In the physical case, $d = 3$ and $D = 2$, of course. Throughout the remainder of this paper, we will distinguish between D-dimensional "intrinsic" vectors and d-dimensional "extrinsic" vectors by using boldface type for the former, and vector arrows over the latter.

We now construct the Landau-Ginzburg-Wilson free energy F for this system, by expanding F to leading order in powers of $\vec{r}(\mathbf{x})$ and its gradients with respect to internal space $\mathbf{x}$, keeping only those terms consistent with the symmetries of the problem. These symmetries are global translation invariance $\vec{r}(\mathbf{x}) \rightarrow \vec{r}(\mathbf{x}) + \vec{r}_o$, and global rotational invariance $\vec{r}(\mathbf{x}) \rightarrow \overleftrightarrow{M} \cdot \vec{r}(\mathbf{x})$, where $\vec{r}_o$ and $\overleftrightarrow{M}$ are a constant (i.e. $\mathbf{x}$-independent) vector and a constant rotation matrix, respectively. Global translational invariance requires that F be expanded only in powers of *gradients* with respect to $\mathbf{x}$. We will furthermore take the membrane to be isotropic in the $D - 1$ membrane directions (hereafter denoted by $\mathbf{x}_\perp$) orthogonal to one special direction (which we call y). Since the physical case is $D = 2$, this specialization is innocuous.

The most general model consistent with all of these symmetries, neglecting irrelevant terms, is,

$$F[\vec{r}(\mathbf{x})] = \frac{1}{2} \int d^{D-1}x_\perp dy \left[\kappa_\perp \left(\partial_\perp^2 \vec{r} \right)^2 + \kappa_y \left(\partial_y^2 \vec{r} \right)^2 \right.$$
$$+ \kappa_{\perp y} \partial_y^2 \vec{r} \cdot \partial_\perp^2 \vec{r} + t_\perp \left(\partial_\alpha^\perp \vec{r} \right)^2 + t_y \left(\partial_y \vec{r} \right)^2$$
$$+ \frac{u_{\perp\perp}}{2} \left(\partial_\alpha^\perp \vec{r} \cdot \partial_\beta^\perp \vec{r} \right)^2 + \frac{u_{yy}}{2} \left(\partial_y \vec{r} \cdot \partial_y \vec{r} \right)^2 + u_{\perp y} \left(\partial_\alpha^\perp \vec{r} \cdot \partial_y \vec{r} \right)^2$$
$$\left. + \frac{v_{\perp\perp}}{2} \left(\partial_\alpha^\perp \vec{r} \cdot \partial_\alpha^\perp \vec{r} \right)^2 + v_{\perp y} \left(\partial_\alpha^\perp \vec{r} \right)^2 \left(\partial_y \vec{r} \right)^2 \right]$$
$$+ \frac{b}{2} \int d^D x \int d^D x' \ \delta^{(d)} \left(\vec{r}(\mathbf{x}) - \vec{r}(\mathbf{x}') \right) , \qquad (2.1)$$

where the κ's, t's, u's, v's are elastic constants. The first three terms in F (the κ terms) represent the anisotropic bending energy of the membrane. The elastic constants $t_\perp$ and t_y are the most strongly temperature dependent parameters in the model, changing sign from large, positive values at high temperatures to negative values at low temperatures. Their positivity at high temperatures reflects the membrane's entropic preference for crumpling. To see this, note that this crumpled state is one in which all the particles in the membrane attempt to cram themselves into the same point $\vec{r}$; in this state, the gradients with respect to the internal space $\partial_\alpha^\perp \vec{r}$ and $\partial_y \vec{r}$ seek to minimize themselves, which is clearly favorable when $t_\perp$, $t_y > 0$. However, when either of these becomes negative, it becomes favorable for the membrane to flatten (i.e., extend) in the associated direction, as we shall show in a moment. The u and v quartic terms are higher order elastic constants needed to stabilize the membrane when one or both of the first order elastic constants $t_\perp$, t_y become negative. Stability requires that

$$u'_{\perp\perp} > 0 , \tag{2.2}$$

$$u_{yy} > 0 , \tag{2.3}$$

$$v_{\perp y} > -\sqrt{u'_{\perp\perp} u_{yy}} , \tag{2.4}$$

where

$$u'_{\perp\perp} \equiv v_{\perp\perp} + u_{\perp\perp}/(D-1) . \tag{2.5}$$

The final, b term in Eq.2.1 represents the self-avoidance of the membranes; i.e., its steric or excluded volume interaction.

Equation 2.1 reduces to the model for isotropic membranes considered in Ref. 24 when $t_\perp = t_y$, $\kappa_{\perp\perp} = \kappa_y$, $\kappa_{\perp y} = 0$, $u_{yy} = 4(\tilde{v}+u)$, $u_{\perp\perp} = u_{\perp y} = 4u$, and $v_{\perp\perp} = v_{\perp y} = 4\tilde{v}$.

III. MEAN FIELD THEORY

We begin our analysis of this model by obtaining its mean field phase diagram, at first neglecting the self-avoidance interaction. Later, we will consider both the effects of fluctuations and self-avoidance.

In mean-field theory, we seek a configuration $\vec{r}(\mathbf{x})$ that minimizes the free energy Eq.2.1 (without the self-avoidance term). The curvature energies $\kappa_\perp \left(\partial_\perp^2 \vec{r}\right)^2$ and $\kappa_y \left(\partial_y^2 \vec{r}\right)^2$ are clearly minimized when $\vec{r}(\mathbf{x})$ is linear in $\mathbf{x}$. We will therefore seek minima of F of the form

$$\vec{r}(\mathbf{x}) = (\zeta_\perp \mathbf{x}_\perp, \zeta_y y, 0, 0, \ldots, 0) . \tag{3.1}$$

Obviously, uniform rotations $\vec{r}(\mathbf{x}) \to \overset{\leftrightarrow}{M} \cdot \vec{r}(\mathbf{x})$, of any such minimum, with $\overset{\leftrightarrow}{M}$ a constant rotation matrix, will also be minima. A continuous degenerate set of minima is thereby obtained, as usual for a system with a broken continuous symmetry. Uniform translations of the entire membrane are also allowed, of course.

Inserting Eq.3.1 into Eq.2.1, and for now neglecting the self-avoidance term, we obtain the mean-field free energy for anisotropic membranes

$$F = \frac{1}{2} L_\perp^{D-1} L_y \left[t_y \zeta_y^2 + t_\perp (D-1)\zeta_\perp^2 \right.$$
$$\left. + \frac{1}{2} u'_{\perp\perp}(D-1)^2 \zeta_\perp^4 + \frac{1}{2} u_{yy}\zeta_y^4 + v_{\perp y}(D-1)\zeta_\perp^2 \zeta_y^2 \right] , \tag{3.2}$$

where $L_\perp$ and L_y are the linear dimensions of the flattened membrane in the $\perp$ and y directions, respectively.

This mean field theory is precisely that studied long ago by Fisher et al.[25] for a completely different (magnetic) problem. Minimizing the free energy over $\zeta_\perp$ and ζ_y yields two possible phase diagram topologies, depending on whether $u'_{\perp\perp} u_{yy} > v_{\perp y}^2$ or $u'_{\perp\perp} u_{yy} < v_{\perp y}^2$.

For $u'_{\perp\perp} u_{yy} > v_{\perp y}^2$, we obtain the phase diagram in Fig.1. Both $\zeta_\perp$ and ζ_y vanish for $t_\perp, t_y > 0$. This is the

crumpled phase: the entire membrane, in mean-field theory, collapses into the origin, $\zeta_\perp = \zeta_y = 0$ i.e., $\vec{r}(\mathbf{x}) = 0$ for all $\mathbf{x}$.

In the regime between the positive $t_\perp$-axis (i.e., the locus $t_y = 0$ and $t_\perp > 0$) and the $t_y < 0$ part of the $t_y = (u_{yy}/v_{\perp y})t_\perp$ line, lies our new y-tubule phase, characterized by $\zeta_\perp = 0$ and $\zeta_y = \sqrt{|t_y|/u_{yy}} > 0$: the membrane is extended in the y-direction but crumpled in all $D-1$ $\perp$-directions.

The $\perp$-tubule phase is the analogous phase with the y and $\perp$ directions reversed, $\zeta_y = 0$ and $\zeta_\perp = \sqrt{|t_\perp|/u_{\perp\perp}} > 0$ (obviously a symmetrical reversal for the physical case of $D=2$), and lies between the $t_\perp < 0$ segment of the line $t_y = (v_{\perp y}/u'_{\perp\perp})t_\perp$ and the positive t_y axis. Finally, the flat phase, characterized by both

$$\zeta_\perp = \left[(|t_\perp|u_{yy} - |t_y|v_{\perp y})/(u'_{\perp\perp}u_{yy} - v_{\perp y}^2) \right]^{1/2} > 0 , \tag{3.3}$$

$$\zeta_y = \left[(|t_y|u_{\perp\perp} - |t_\perp|v_{\perp y})/(u'_{\perp\perp}u_{yy} - v_{\perp y}^2) \right]^{1/2} > 0 , \tag{3.4}$$

lies between the $t_\perp < 0$ segment of the line $t_y = (u_{yy}/v_{\perp y})t_\perp$ and the $t_y < 0$ segment of the line $t_y = (v_{\perp y}/u'_{\perp\perp})t_\perp$.

For $u'_{\perp\perp} u_{yy} < v_{\perp y}^2$, the flat phase disappears, and is replaced by a direct first-order transition from $\perp$-tubule to y-tubule along the locus $t_y = (v_{\perp y}/u'_{\perp\perp})t_\perp$ (see Fig.4) .
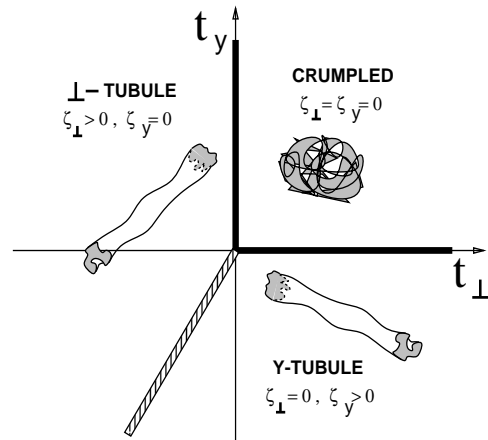

FIG. 4. Phase diagram for tethered membranes showing our new tubule phase, for the range of elastic parameters when the intermediate flat phase disappears. A first-order phase transition separates y- and $\perp$-tubule phases.

The boundaries between the tubule and the crumpled phases remain the positive t_y and $t_\perp$ axes, as for $u'_{\perp\perp} u_{yy} > v_{\perp y}^2$ case.

Note that a direct crumpling transition (i.e. a direct transition between the crumpled and flat phase) is very non-generic in this picture: only experimental loci that pass from $t_y, t_\perp > 0$ through the origin (locus P_2 in Fig.1) can experience such a transition. This transition is, in fact, tetra-critical in this picture.

This does not, however, imply that direct crumpling transitions are non-generic. Many membranes will be

perfectly isotropic, by virtue of being formed under conditions of unbroken rotational symmetry (e.g., randomly polymerized membranes). As discussed earlier, this set of membranes, which is undoubtedly of finite measure, necessarily lies on the special isotropic subspace of the full parameter space of the model defined by Eq.2.1 specified by $t_\perp = t_y$, $\kappa_{\perp\perp} = \kappa_y$, $\kappa_{\perp y} = 0$, $u_{yy} = 4(\tilde{v} + u)$, $u_{\perp\perp} = u_{\perp y} = 4u$, and $v_{\perp\perp} = v_{\perp y} = 4\tilde{v}$. The values of the quartic couplings then satisfy $u'_{\perp\perp} u_{yy} > v_{\perp y}^2$ (for u, $\tilde{v} > 0$), and hence the topology of the phase diagram is Figure 1. The boundaries of the flat phase for those isotropic values of the quartic couplings become $t_y = t_\perp (1 + u/\tilde{v}/(D-1))$ and $t_y = (1 + u/\tilde{v}) t_\perp$, respectively. For u and v both positive (as required by stability), the slopes of these lines are less than and greater than 1, respectively; the isotropic locus $t_y = t_\perp$ therefore lies between the two (i.e., in the flat phase), and hence, that model *does* undergo a direct flat to crumpled transition.

Membranes with *any* intrinsic broken orientational symmetry (e.g., in-plane tilt order[9], which is quite common[12]), will generically have $t_y \neq t_\perp$. Furthermore, they will not generically have both $t_\perp$ and t_y vanish at the same temperature. A generic locus through the phase diagram in Fig.1 will be like locus P_1, and will necessarily have one of the tubule phases intervening between the flat and crumpled phases. Our new tubule phase is not only generically possible, but actually unavoidable, in membranes with any type or amount of *intrinsic anisotropy*.

IV. FLUCTUATIONS AND SELF-AVOIDANCE IN THE FLAT AND CRUMPLED PHASES

In this section, we show that both the flat and the crumpled phases of anisotropic membranes are identical in their scaling properties, at sufficiently long length scales, to the eponymous phases of isotropic membranes.

Consider first the flat phase. We can include fluctuations about the mean-field solution by considering small deviations from the solution in Eq.3.1

$$\vec{r}(\mathbf{x}) = \left(\zeta_\perp \mathbf{x}_\perp + u_\perp(\mathbf{x}), \zeta_y y + u_y(\mathbf{x}), \vec{h}(\mathbf{x}) \right) , \quad (4.1)$$

Inserting this into our initial free energy, Eq.2.1, with $t_\perp$ and t_y both in the range in which the flat phase is stable, we obtain the uniaxial elastic energy of Ref. 26. As shown in that reference, fluctuation effects in turn renormalize the anisotropic elastic energy into the *isotropic* membrane elastic energy considered by Refs. 5,7,8. In the flat phase, and at sufficiently long scales, the anisotropic membranes therefore behave exactly like isotropic membranes. This in particular implies that the flat phase of anisotropic membranes is stable against thermal fluctuations. As in isotropic membranes, this is due to the fact that these very thermal fluctuations drive the bend modulus κ to infinity at long wavelengths[5,7,8].

Specifically, κ becomes wavevector dependent, and $\kappa(\mathbf{q})$ diverges like $q^{-\eta_\kappa}$ as $q \to 0$. In the flat phase the standard Lamé coefficients μ and λ [27] are also infinitely renormalized and become wavevector dependent, vanishing in the $q \to 0$ limit as $\mu(q) \sim \lambda(q) \sim q^{\eta_u}$; the values of η_κ and η_u in the flat phase differ from those in the tubule phase, as does their physical interpretation. The flat phase is furthermore novel in that it is characterized by a universal *negative* Poisson ratio[7,28] which for $D = 2$ is defined as the long wavelength limit $q \to 0$ of $\sigma = \lambda(q)/(2\mu(q) + \lambda(q))$. The transverse undulations in the flat phase, i.e. the membrane roughness h_{rms} scales with the internal size of the membrane as $h_{rms} \sim L^\zeta$, with $\zeta = (4 - D - \eta_\kappa)/2$, exactly. Furthermore, an underlying rotational invariance imposes an exact Ward identity between η_κ and η_u, $\eta_u + 2\eta_\kappa = 4 - D$, leaving only a single nontrivial independent exponent characterizing the properties of the flat phase of even anisotropic membranes. The best estimate for η_κ in the physical case of a two-dimensional membrane ($D = 2$), embedded in a $d = 3$-dimensional space comes from the self-consistent screening approximation (SCSA) of Le Doussal and Radzihovsky[28], who find $\eta_\kappa = 4/(1 + \sqrt{15}) \approx 0.82$. The exponent relations above then predict $\eta_u = 0.36$ and $\zeta = 0.59$. These exponents, together with the negative Poisson ratio predictions of Le Doussal and Radzihovsky of $\sigma = -1/3$ [28] have been recently spectacularly verified to high precision in very large scale simulations (largest to date) by Falcioni, Bowick, Guitter and Thorleifsson[29].

The root-mean-square (rms) thermal fluctuation $\langle |\hat{n}(\mathbf{x}) - \hat{z}|^2 \rangle \equiv \langle |\delta\vec{n}(\mathbf{x})|^2 \rangle$ of the local membrane normal $\hat{n}(\mathbf{x})$ about its mean value (here taken to be $\hat{z}$) is

$$
\begin{aligned}
\langle |\delta\vec{n}(\mathbf{x})|^2 \rangle &= \langle |\boldsymbol{\nabla}\vec{h}(\mathbf{x})|^2 \rangle , \\
&= \int d^D q \, q^2 \langle |\vec{h}(\mathbf{q})|^2 \rangle , \\
&\propto \int \frac{d^D q}{\kappa(q)q^2} \propto \int \frac{d^D q}{q^{2-\eta_\kappa}} , \\
&\propto L^{2-\eta_\kappa - D} , \quad (4.2)
\end{aligned}
$$

where we have imposed an infra-red cutoff $q > L^{-1}$, on the integral over wavevectors, L being the smaller of the intrinsic linear dimensions $L_\perp$, L_y of the flattened membrane. These fluctuations are finite as $L \to \infty$, when $2 - \eta_\kappa - D < 0$. In the physical case $D = 2$, this condition is always satisfied since $\eta_\kappa > 0$. Thus, membrane orientational fluctuations remain bounded, and the flat phase is stable against thermal fluctuations, for the physical case $D = 2$. Indeed, the SCSA predicts that they remain bounded down to the lower critical dimension $D = \sqrt{2}$. [28]

Note that this stability of the flat phase depends crucially on the anomalous elasticity, i.e., the divergence of $\kappa(q)$ as $q \to 0$. In the absence of this effect, which would correspond to $\eta_\kappa = 0$, the integral over wavevector in

Eq.4.2 would diverge logarithmically for $D = 2$, describing divergent orientational fluctuations leading to an instability of the flat phase at any non-zero temperature. Hence, the flat phase owes its stability to the anomalous elasticity (i.e., the fact that $\eta_\kappa > 0$). In contrast, as we shall show in a moment, the tubule phase is marginally *stable* against thermal fluctuations, even in the absence of anomalous elastic effects. Such effects are, nonetheless, actually present for self-avoiding tubules, but they are not essential to the stability of the phase.

Because of this persistent long-ranged orientational order (i.e., because the membrane is flat), widely intrinsically separated parts of the membranes (i.e., points $\mathbf{x}$ and $\mathbf{x}'$, with $|\mathbf{x} - \mathbf{x}'|$ large) do not bump into each other (i.e., never have $\vec{r}(\mathbf{x}) = \vec{r}(\mathbf{x}')$); hence, the self-avoidance interaction in Eq.2.1 is irrelevant in the flat phase.

That the crumpled phase of anisotropic membranes is identical to that of isotropic membranes is even easier to see. When both $t_\perp$ and t_y are positive, all of the other local terms in Eq.2.1, i.e., the κ, u, and v-terms, are irrelevant at long wavelengths (since they all involve more derivatives than the t-terms). Once these irrelevant terms are neglected, a simple change of variables $\mathbf{x}_\perp = \mathbf{x}'\sqrt{t_\perp/t_y}$ makes the remaining energy isotropic. Thus, the entire crumpled phase is identical in its scaling properties to that of isotropic membranes.

In particular, the membrane in this phase has a radius of gyration $R_G(L)$ which scales with membrane linear dimension L like L^ν, with $\nu = (D+2)/(d+2)$ in Flory theory, and very similar values predicted by ϵ-expansion techniques[30−32].

V. FLUCTUATIONS IN PHANTOM TUBULES

In this section, we ignore self-avoidance (i.e., treat "phantom" membranes), and consider the effects of fluctuations on phantom tubules. We will show that these fluctuations do not destroy the tubule phase, or change the topology of the phase diagram. The detailed properties of the tubule phase are, however, modified by the fluctuations.

Let us consider the y-tubule phase (i.e., the tubule phase with the tubule axis along the y-axis). To treat fluctuations, we perturb around the mean-field solution $\vec{r}_o(\mathbf{x}) = \zeta_y(y, \vec{0})$ by writing

$$\vec{r}(\mathbf{x}) = \left(\zeta_y y + u(\mathbf{x}), \vec{h}(\mathbf{x}) \right) , \qquad (5.1)$$

where $\vec{h}(\mathbf{x})$ is a $d - 1$-component vector orthogonal to the tubule's axis, which we take to be oriented along the y-axis. The average extension factor ζ_y is near but not exactly equal to its mean-field value, because fluctuations will change it. Rather, we will choose ζ_y so that all linear terms in $\vec{h}(\mathbf{x})$ and $u(\mathbf{x})$ in the resultant elastic free energy for these variables are exactly cancelled, in the long wavelength limit, by their fluctuation renormalizations.

This criterion guarantees that $\vec{h}(\mathbf{x})$ and $u(\mathbf{x})$ represent fluctuations around the true ground state of F. Precisely analogous choices have been used in the study of bulk smectic A elasticity[15], and the flat-phase elasticity of isotropic membranes[5,7,8].

Inserting the decomposition Eq.5.1 into the free energy Eq.2.1, neglecting irrelevant terms, and, for the moment ignoring the self-avoidance interaction, gives, after some algebra, the elastic free energy $F_{tot} = F_{mft} + F_{el}$, where F_{mft} is simply the mean-field free energy for the tubule phase

$$F_{mft} = \frac{1}{2} L_\perp^{D-1} L_y \left[t_y \zeta_y^2 + \frac{1}{4} u_{yy} \zeta_y^4 \right] , \qquad (5.2)$$

and $F_{el}[u(\mathbf{x}), \vec{h}(\mathbf{x})]$ is the fluctuating elastic free energy part

$$F_{el} = \frac{1}{2} \int d^{D-1} x_\perp dy \left[\gamma \left(\partial_y u + \frac{1}{2} (\partial_y \vec{h})^2 + \frac{1}{2} (\partial_y u)^2 \right) \right.$$
$$+ \kappa (\partial_y^2 \vec{h})^2 + t (\partial_\alpha^\perp \vec{h})^2 + g_\perp (\partial_\alpha^\perp u)^2$$
$$\left. + g_y \left(\partial_y u + \frac{1}{2} (\partial_y \vec{h})^2 + \frac{1}{2} (\partial_y u)^2 \right)^2 \right] , \qquad (5.3)$$

where $\kappa \equiv \kappa_y$, $t \equiv t_\perp + v_{\perp y} \zeta_y^2$, $g_y \equiv u_{yy} \zeta_y^2/2$, $g_\perp \equiv t + u_{\perp y} \zeta_y^2$, and $\gamma = t_y + u_{yy} \zeta_y^2$ are constant coefficients. Note first that the coefficient γ of the linear terms in F_{el} is also the coefficient of the $(\partial_y \vec{h})^2$ term. This is a consequence of the rotation invariance of the original free energy Eq.2.1, which leads to the existence of the Goldstone mode $\partial_y \vec{h}$. The combination $E(u, \vec{h}) \equiv \partial_y u + \frac{1}{2}(\partial_y \vec{h})^2 + \frac{1}{2}(\partial_y u)^2$ is the only combination of first y-derivatives of u and $\vec{h}$ that is invariant under global rotations of the tubule. It is analogous to the nonlinear strain tensor of conventional elasticity theory.[27] On these general symmetry grounds, therefore, the free energy can only depend on $\partial_y u$ and $\partial_y \vec{h}$ through powers of $E(u, \vec{h})$, and this property must be preserved upon renormalization. This has two important consequences: the first is that, since, as discussed earlier, the coefficient of this linear term will be chosen to vanish upon renormalization via a judicious choice of the stretching factor ζ_y, the coefficient of $(\partial_y \vec{h})^2$ will likewise vanish.[33] This means that the y-direction becomes a "soft" direction for fluctuations of $\vec{h}$ in the tubule phase. We can trace this softness back to the spontaneously broken rotational symmetry of the tubule state. It is precisely analogous to the softness of height fluctuations in the flat phase of isotropic membranes, manifested by the absence of $(\partial_x \vec{h})^2$, $(\partial_y \vec{h})^2$ terms in the elastic free energy of the flat phase, analogous to Eq.5.3 (when γ is tuned to 0).

The second important consequence is that the ratios of the coefficients of the quadratic $(\partial_y u)^2$ and the anharmonic $\partial_y u(\partial_y \vec{h})^2$ and $(\partial_y \vec{h})^4$ terms in F_{el} must always be *exactly* $4 : 4 : 1$, since they must appear together as

a result of expanding $(\partial_y u + \frac{1}{2}(\partial_y \vec{h})^2 + \frac{1}{2}(\partial_y u)^2)^2$. We will show in a few moments that, for this special value of these ratios, the long-wavelength anomalous elastic behavior of the "phantom" tubule phase can be calculated *exactly*.

Recognizing that γ vanishes after renormalization, we can now calculate the propagators (i.e., the harmonic approximation to the Fourier transformed correlation functions) by setting $\gamma = 0$ in Eq.5.3. We thereby obtain

$$\langle h_i(\mathbf{q}) h_j(-\mathbf{q}) \rangle = k_B T \delta_{ij}^{\perp} G_h(\mathbf{q}) , \qquad (5.4)$$

$$\langle u(\mathbf{q}) u(-\mathbf{q}) \rangle = k_B T G_u(\mathbf{q}) , \qquad (5.5)$$

where

$$G_h^{-1}(\mathbf{q}) = tq_{\perp}^2 + \kappa q_y^4 , \qquad (5.6)$$

$$G_u^{-1}(\mathbf{q}) = g_{\perp} q_{\perp}^2 + g_y q_y^2 , \qquad (5.7)$$

and $\delta_{ij}^{\perp}$ is a Kronecker delta when both indices i and $j \neq y$, and is zero if either i or $j = y$.

Inspection of the propagators G_h and G_u reveals that the $\vec{h}$-fluctuations are much larger than the u-fluctuations for $|\mathbf{q}_{\perp}| \approx q_y^2$, and that it is precisely this regime of wavevectors that dominates the fluctuations. Thus, in power counting to determine the relevance or irrelevance of various operators, we must count each power of $|\mathbf{q}_{\perp}|$ as *two* powers of q_y. It is this power counting that leads to the identification of the terms explicitly displayed in Eq.5.3 as the most relevant ones.

Calculating the root-mean-squared (rms) real space positional fluctuations $\langle |\vec{h}(\mathbf{x})|^2 \rangle$ in the harmonic approximation by integrating the propagators over all wavevectors, we find

$$\langle |\vec{h}(\mathbf{x})|^2 \rangle \propto \int_{q_{\perp} > L_{\perp}^{-1}} \frac{d^{D-1} q_{\perp} dq_y}{(2\pi)^D} \frac{1}{tq_{\perp}^2 + \kappa q_y^4} ,$$

$$\propto \int_{q_{\perp} > L_{\perp}^{-1}} \frac{d^{D-1} q_{\perp}}{q_{\perp}^{3/2}} \propto L_{\perp}^{5/2-D} , \qquad (5.8)$$

where we have introduced an infra-red cutoff $|\mathbf{q}_{\perp}| > L_{\perp}^{-1}$ in the last integral. This expression clearly reveals that for "phantom" tubules, the upper critical dimension D_{uc} for this problem, below which transverse positional fluctuations diverge is $D_{uc} = 5/2$; this in principle (but see discussion of dominant zero modes in Sec.VI B) allows a quantitatively trustworthy $\epsilon = D_{uc} - 2 = 1/2$ expansion for the physical membrane of $D = 2$. This should be contrasted with the result $D_{uc} = 4$ for the analogous critical dimension in the flat phase.[7,8]

The lower critical dimension D_{lc} below which the tubule is necessarily crumpled in this problem is also lowered by the anisotropy. Considering the fluctuations of the membrane normals $\boldsymbol{\nabla} \vec{h}$ in the harmonic approximation, one sees immediately that the largest of these is the fluctuation in the y-direction,

$$\langle |\delta n_y(\mathbf{x})|^2 \rangle = \langle |\partial_y \vec{h}(\mathbf{x})|^2 \rangle ,$$

$$\propto \int_{q_{\perp} > L_{\perp}^{-1}} \frac{d^{D-1} q_{\perp} dq_y}{(2\pi)^D} \frac{q_y^2}{tq_{\perp}^2 + \kappa q_y^4} ,$$

$$\propto \int_{q_{\perp} > L_{\perp}^{-1}} \frac{d^{D-1} q_{\perp}}{q_{\perp}^{1/2}} \propto L_{\perp}^{3/2-D} , \qquad (5.9)$$

which clearly only diverges in the infra-red $L_{\perp} \to \infty$ limit for $D \leq D_{lc} = 3/2$ (but again see discussion of dominant zero modes in Sec.VI B).

In the argot of the membrane field, the elasticity of phantom tubules is anomalous. In contrast to the flat phase, however, for phantom tubules, the exponents characterizing the anomalous elasticity can be calculated *exactly*. To see this, we first note that the u-fluctuations go like $1/q^2$ in *all* directions and hence are negligible (in the relevant wavevector regime $|\mathbf{q}_{\perp}| \approx q_y^2$) relative to the $\vec{h}$-fluctuations which scale like $1/q^4$ in this regime. This justifies neglecting the $\frac{1}{2}(\partial_y u)^2$ piece of the invariant $E(u, \vec{h})$ operator. This also emerges from a full renormalization group treatment[33], which shows that this term is strongly irrelevant. Once it is neglected, the elastic free energy is quadratic in u, and these phonon modes can therefore be integrated exactly out of the partition function

$$Z = \int \mathcal{D}u \mathcal{D}\vec{h} e^{-\beta F_{el}[u, \vec{h}]} . \qquad (5.10)$$

Once this is done, the only remaining anharmonic term in the effective elastic free energy for $\vec{h}$ is, in Fourier space,

$$F_{anh}[\vec{h}] = \frac{1}{4} \int_{\mathbf{k}_1, \mathbf{k}_3, \mathbf{k}_3} \left(\vec{h}(\mathbf{k}_1) \cdot \vec{h}(\mathbf{k}_2) \right) \left(\vec{h}(\mathbf{k}_3) \cdot \vec{h}(\mathbf{k}_4) \right) \times$$

$$k_{y1} k_{y2} k_{y3} k_{y4} V_h(\mathbf{q}) , \qquad (5.11)$$

where $\mathbf{q} = \vec{k}_1 + \vec{k}_2$ and $\vec{k}_1 + \vec{k}_2 + \vec{k}_3 + \vec{k}_4 = 0$. The effective vertex $V_h(\mathbf{q})$ above reduces to

$$V_h(\mathbf{q}) = \frac{g_y g_{\perp} q_{\perp}^2}{g_y q_y^2 + g_{\perp} q_{\perp}^2} , \qquad (5.12)$$

which is irrelevant near the Gaussian fixed point (but see Sec.VI B), as can be seen by the simple *anisotropic* power counting described above.

The exact cancelation of the relevant terms in $F_{anh}[\vec{h}]$ above is a direct consequence of the $4 : 4 : 1$ ratios of the coefficients of the quadratic $(\partial_y u)^2$ and the anharmonic $\partial_y u (\partial_y \vec{h})^2$ and $(\partial_y \vec{h})^4$ terms in F_{el} that was discussed earlier. Given this cancelation, $F_{anh}[\vec{h}]$ is now clearly less relevant than the anharmonic vertices $\partial_y u (\partial_y \vec{h})^2$ and $(\partial_y \vec{h})^4$ in the original free energy (before we integrated out the phonons u). This is because the factor $V_h(\mathbf{q}) \propto q_{\perp}^2 / (g_y q_y^2 + g_{\perp} q_{\perp}^2)$ vanishes like q_y^2 in the relevant limit $|\mathbf{q}_{\perp}| \sim q_y^2$, $q_y \to 0$ (the other factors in Eq.5.11 are precisely the Fourier transform of $(\partial_y \vec{h})^4$, of course). This lowers the upper critical dimension for anomalous

elasticity *of the $\vec{h}$ field* to $D_{uc} = 3/2$. Thus, in the physical case $D = 2$, there is no anomalous elasticity in $\vec{h}$; that is, the elastic constants t and κ in Eq.5.6 are finite and non-zero as $q_y \to 0$.

However, as asserted earlier, the *full* elasticity Eq.5.3, *before u is integrated out, is* anomalous, because g_y is driven to zero as $q_y \to 0$. Indeed, a self-consistent one-loop perturbative calculation of $g_y(\mathbf{q})$, obtained by evaluating the Feynman graph in Fig.5, gives

$$g_y(\mathbf{q}) = g_y^o - \qquad\qquad (5.13)$$
$$\int \frac{k_B T g_y^2(\mathbf{q}) p_y^2 (p_y - q_y)^2 \; d^{D-1}p_\perp dp_y/(2\pi)^D}{\left(t p_\perp^2 + \kappa(\mathbf{p}) p_y^4\right)\left(t|\mathbf{p} - \mathbf{q}_\perp|^2 + \kappa(|\mathbf{p} - \mathbf{q}|)(p_y - q_y)^4\right)} \; ,$$

where g_y^o is the "bare" or unrenormalized value of g_y.
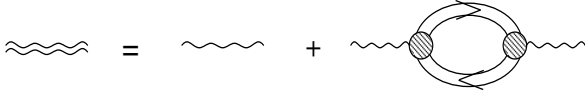
FIG. 5. Feynman graph equation for the self-consistent evaluation of $g_y(\mathbf{q})$.

Our earlier argument shows that $\kappa(\mathbf{p})$ can be replaced by a constant in Eq.5.13 as $\mathbf{p} \to 0$, since the $\vec{h}$ elasticity is not anomalous. The self-consistent equation Eq.5.13 can be solved by the ansatz,

$$g_y(\mathbf{q}) = q_y^{\eta_u} S_g(q_y/q_\perp^z) \; . \qquad\qquad (5.14)$$

Simple power counting[34] then shows that we must choose

$$z = \frac{1}{2} \; , \qquad\qquad (5.15)$$
$$\eta_u = 5 - 2D \; . \qquad\qquad (5.16)$$

It is straightforward to verify that these results hold to *all* orders in perturbation theory; that is, at every order, the leading dependence on $\mathbf{q}$ of the contribution to g_y scales like $q_y^{\eta_u} S_g(q_y/q_\perp^{1/2})$ with $\eta_u = 5 - 2D$.

It is straightforward to verify to *all* orders in perturbation theory that there is no such renormalization of $g_\perp$. This is because of the anisotropic scaling $q_\perp \sim q_y^2$, which implies that all vertices proportional to powers of *perpendicular* gradients of $\vec{h}$, i.e., powers of $\boldsymbol{\nabla}_\perp \vec{h}$ are *irrelevant*. Since only such vertices can renormalize $g_\perp |\boldsymbol{\nabla}_\perp u|^2$, there are no *relevant* renormalization of $g_\perp$. As a result, $g_\perp$ remains finite and non-zero, or, in a word, non-anomalous, as $|\mathbf{q}| \to 0$.

Using the facts that $g_y(\mathbf{q})$ is independent of $\mathbf{q}_\perp$ as $|\mathbf{q}_\perp| \equiv q_\perp \to 0$ for fixed q_y, and, likewise, to be independent of q_y as $q_y \to 0$ for fixed $q_\perp$, we can obtain the limits of the scaling function $S_g(x)$:

$$S_g(x) \propto \begin{cases} \text{constant}, & x \to \infty \\ x^{-\eta_u}, & x \to 0 \; . \end{cases} \qquad (5.17)$$

For *phantom* membranes with $D = 2$, $\eta_u = 1$ and $z = 1/2$, so we find:

$$g_y(\mathbf{q}) \propto \begin{cases} q_y, & q_y >> \sqrt{q_\perp} \\ \sqrt{q_\perp}, & q_y << \sqrt{q_\perp} \; . \end{cases} \qquad (5.18)$$

We will now use this result to compute the mean-squared real space fluctuations $\langle (u(L_\perp, y) - u(0, y))^2 \rangle \equiv \langle \Delta u^2 \rangle$ of $u(\mathbf{x})$. These can be obtained via the equipartition theorem and by summing all of the Fourier modes, yielding:

$$\langle \Delta u^2 \rangle \approx \int_{q_\perp > L_\perp^{-1}, \; q_y > L_y^{-1}} \frac{dq_\perp dq_y}{(2\pi)^2} \frac{1 - e^{i\mathbf{q}_\perp \cdot \mathbf{L}_\perp}}{g_y(\mathbf{q}) q_y^2 + g_\perp q_\perp^2} \; .$$
$$(5.19)$$

Let us assume, and verify a posteriori, that the integral in this expression is dominated by wavevectors with $q_y << \sqrt{q_\perp}$. Then, using Eq.5.18, we see that

$$\langle \Delta u^2 \rangle \approx \int_{q_\perp > L_\perp^{-1}, \; q_y > L_y^{-1}} \frac{dq_\perp dq_y}{(2\pi)^D} \frac{1 - e^{i\mathbf{q}_\perp \cdot \mathbf{L}_\perp}}{c\sqrt{q_\perp} q_y^2 + g_\perp q_\perp^2} \; ,$$
$$(5.20)$$

where c is a constant. Inspection of this integral reveals that it is dominated by $\mathbf{q}$'s for which the two terms in the denominator balance; this means $q_y \sim q_\perp^{3/4} << \sqrt{q_\perp}$, the last extreme inequality holding as $|\mathbf{q}| \to 0$. This verifies our earlier a posteriori assumption that $q_y << \sqrt{q_\perp}$ in the dominant wavevector regime.

Now, changing variables in the integral $q_\perp \equiv Q_\perp/L_\perp$, $q_y \equiv Q_y/L_\perp^{3/4}$, we find

$$\langle \Delta u^2 \rangle = L_\perp^{1/4} S_u(L_y/L_\perp^{3/4}) \; , \qquad (5.21)$$

where

$$S_u(x) \equiv \int_{Q_\perp > 1, \; Q_y > x^{-1}} \frac{dQ_\perp dQ_y}{(2\pi)^2} \frac{1 - e^{iQ_\perp}}{c\sqrt{Q_\perp} Q_y^2 + g_\perp Q_\perp^2} \; .$$
$$(5.22)$$

We note that the scaling form for the u phonon correlations is *different* than that of the height field $\vec{h}$ as summarized in e.g. Eqs.1.1 and 1.2, and discussed in more detail below.

The limits of $S_u(x)$ scaling function can be obtained just as we did for $S_g(x)$; we find, including "zero modes" (see below):

$$S_u(x) \propto \begin{cases} \text{constant}, & x \to \infty \\ x^{-1}, & x \to 0 \; . \end{cases} \qquad (5.23)$$

For roughly square membranes, $L_y \sim L_\perp = L$, so, as $L \to \infty$, $L_y/L_\perp^{3/4} \to \infty$, and the first limit of Eq.5.23 is the appropriate one. This gives

$$\langle \Delta u^2 \rangle \propto L_\perp^{1/4} \ . \tag{5.24}$$

The authors of Ref. 2 measured a quantity that should scale like $\langle \Delta u^2 \rangle$ in their simulations of a square anisotropic membrane. They did this via their vividly named "salami" method: measuring the moment of inertia tensor of a "salami" slice, a set of N points that all had the same internal y coordinate (for a y-tubule phase). It is straightforward to show that the smallest eigenvalue of this tensor should scale like $N\langle u(\mathbf{x})^2 \rangle$, since, as we shall see in a moment, the mean squared displacements in the other directions are much bigger than those in the y direction. Therefore, from Eq.5.24 we predict that the smallest eigenvalue of this salami slice moment of inertia tensor scales like $NL^{1/4}$. BFT actually fit this eigenvalue to $N \log L$, which might appear to disagree with our prediction, until one recognizes that for L's between 32 and 100 (where most of the data of Ref. 2 is taken), $L^{1/4} = (e/4) \log L$ to an accuracy of better than 1%. Thus, their fit is certainly consistent with our predictions. To more strenuously test our full scaling predictions Eq.5.21 and 5.23, one could simulate membranes with aspect ratios quite different from 1. In particular, we predict based on Eq.5.21 that increasing L_y at fixed $L_\perp$ from an initially square configuration would *not* increase this smallest eigenvalue; nor would decreasing L_y decrease it, until an aspect ratio $L_y \sim L_\perp^{3/4}$ is reached, beyond which this eigenvalue would increase like L_y^{-1}.

We now turn to the computation, for the phantom tubule, of the tubule radius of gyration R_G and roughness h_{rms}, defined by:

$$R_G^2 \equiv \langle |\vec{h}(\mathbf{L}_\perp, y) - \vec{h}(0_\perp, y)|^2 \rangle \ , \tag{5.25}$$

$$h_{rms}^2 \equiv \langle |\vec{h}(\mathbf{x}_\perp, L_y) - \vec{h}(\mathbf{x}_\perp, 0)|^2 \rangle \ , \tag{5.26}$$

where $\mathbf{L}_\perp$ spans the intrinsic $\perp$ space of the membrane. Because R_G is by definition the root-mean-square (rms) distance between two points at the same y, it is roughly the radius of a typical cross-section of the tubule perpendicular to the tubule axis. Likewise, h_{rms} measures fluctuations between points widely separated *along* the tubule axis; hence, it gives the polymer-like transverse "wandering" of the tubule. See Fig.2 for an illustration of R_G and h_{rms}.

The reason we distinguish between these two quantities is that they scale in different ways with the membrane dimensions $L_\perp$ and L_y, in contrast to one's naive expectations. This happens because there are large contributions to both quantities from "zero modes", by which we mean Fourier modes with either $\mathbf{q}_\perp$ or $q_y = 0$. Those with $\mathbf{q}_\perp = 0$ correspond to polymer-like undulations of the entire tubule. Recognizing the existence of both types of modes, we Fourier decompose $\vec{h}(\mathbf{x})$ as follows:

$$\vec{h}(\mathbf{x}) = \frac{1}{\sqrt{L_\perp^{D-1} L_y}} \sum_{\mathbf{q}} \vec{h}_B(\mathbf{q}) e^{i\mathbf{q}\cdot\mathbf{x}} + \frac{1}{\sqrt{L_y}} \sum_{q_y} \vec{h}_{0y}(q_y) e^{iq_y y}$$

$$+ \frac{1}{\sqrt{L_\perp^{D-1}}} \sum_{\mathbf{q}_\perp} \vec{h}_{0\perp}(\mathbf{q}_\perp) e^{i\mathbf{q}_\perp \cdot \mathbf{x}_\perp} \ , \tag{5.27}$$

where B, $0y$, and $0\perp$ denote "bulk modes" (i.e., modes with neither $\mathbf{q}_\perp$ nor $q_y = 0$), and "zero modes" (i.e., modes with either $\mathbf{q}_\perp$ or $q_y = 0$), respectively. Note that we have chosen different normalizations for the three types of modes. For phantom membranes, we proceed by inserting this Fourier decomposition into the harmonic, $\vec{h}$ dependent piece of the elastic free energy F_{el} (which is justified, since, as shown above, the elasticity for $\vec{h}$ for a *phantom tubule* is not anomalous), obtaining:

$$F_0 = \frac{1}{2} \sum_{\mathbf{q}} (tq_\perp^2 + \kappa q_y^4) |\vec{h}_B(\mathbf{q})|^2 + \frac{1}{2} L_\perp^{D-1} \sum_{q_y} \kappa q_y^4 |\vec{h}_{0y}(q_y)|^2$$

$$+ \frac{1}{2} L_y \sum_{\mathbf{q}_\perp} tq_\perp^2 |\vec{h}_{0\perp}(\mathbf{q}_\perp)|^2) \ , \tag{5.28}$$

Note the explicit presence of the factors of $L_\perp^{D-1}$ and L_y for the 0-modes. Applying equipartition to Eq.5.28, we can obtain the mean squared fluctuations of the Fourier modes:

$$\langle |\vec{h}_B(\mathbf{q})|^2 \rangle = \frac{k_B T(d-D)}{tq_\perp^2 + \kappa q_y^4} \tag{5.29}$$

$$\langle |\vec{h}_{0y}(q_y)|^2 \rangle = \frac{k_B T(d-D)}{L_\perp^{D-1} \kappa q_y^4} \tag{5.30}$$

$$\langle |\vec{h}_{0\perp}(\mathbf{q}_\perp)|^2 \rangle = \frac{k_B T(d-D)}{L_y tq_\perp^2} \tag{5.31}$$

Using these expressions inside Eqs.5.25 and 5.26, and being careful about converting sums on $\mathbf{q}$ into integrals, we get

$$R_G^2 = 2(d-D) \left[\frac{k_B T}{L_y} \int_{L_\perp^{-1}} \frac{d^{D-1}q_\perp}{(2\pi)^{D-1}} \frac{1}{tq_\perp^2} (1 - e^{i\mathbf{q}_\perp \cdot \mathbf{L}_\perp}) \right.$$

$$\left. + k_B T \int_{L_\perp^{-1}, L_y^{-1}} \frac{d^{D-1}q_\perp dq_y}{(2\pi)^D} \frac{(1 - e^{i\mathbf{q}_\perp \cdot \mathbf{L}_\perp})}{tq_\perp^2 + \kappa q_y^4} \right] , \quad (5.32)$$

where the subscripts $L_\perp^{-1}$ and L_y^{-1} denote infra-red cutoffs $|\mathbf{q}_\perp| > L_\perp^{-1}, q_y > L_y^{-1}$, with $L_\perp \equiv |\mathbf{L}_\perp|$.

We observe here that R_G in Eq.5.32 does not receive any contribution from the $\mathbf{q}_\perp = 0$ "zero mode" (i.e., in addition to the bulk mode, R_G receives a contribution only from the $q_y = 0$ "zero mode").

Scaling $L_\perp$ out of both integrals for R_G by the change of variables $\mathbf{Q}_\perp \equiv \mathbf{q}_\perp L_\perp$ and $Q_y \equiv q_y \sqrt{L_\perp}$, we obtain

$$R_G^2 = \left(\frac{C_1 L_\perp^{3-D}}{L_y} + L_\perp^{5/2-D} I_R \left(\frac{L_y}{\sqrt{L_\perp}} \right) \right) , \tag{5.33}$$

where

$$C_1 \equiv 2(d-D)k_B T \int_1 \frac{d^{D-1}q_\perp}{(2\pi)^{D-1}} \frac{(1-e^{i\mathbf{q}_\perp \cdot \hat{\mathbf{L}}_\perp})}{tq_\perp^2} \ . \quad (5.34)$$

is a constant of $O(1)$, and

$$I_R(x) \equiv 2(d-D)k_B T \int_{1,x} \frac{d^{D-1}q_\perp dq_y}{(2\pi)^D} \frac{(1-e^{i\mathbf{q}_\perp \cdot \hat{\mathbf{L}}_\perp})}{tq_\perp^2 + \kappa q_y^4}.$$
$$(5.35)$$

with $\hat{\mathbf{L}}_\perp$ the unit vector along $\mathbf{L}_\perp$. Defining the scaling function

$$S_R(x) \equiv \sqrt{\frac{C_1}{x} + I_R(x)}, \quad (5.36)$$

we see that R_G can be rewritten in the scaling form

$$R_G(L_\perp, L_y) = L_\perp^\nu S_R(L_y/L_\perp^z) \quad (5.37)$$

with, for phantom membranes,

$$\nu = \frac{5-2D}{4} \ , \quad (5.38)$$

$$z = \frac{1}{2} \ . \quad (5.39)$$

We will see later that the scaling form Eq.5.37 continues to apply when self-avoidance is included, but with different values of ν and z, and a different scaling function $S_R(x)$. For phantom membranes, from our explicit expression for the scaling function S_R, we see that it has the limiting forms:

$$S_R(x) \propto \begin{cases} 1/\sqrt{x} & \text{for } x \to 0 \\ \text{constant,} & \text{for } x \to \infty \end{cases} \quad (5.40)$$

In particular, the limiting form as $x \to \infty$ implies that for the physically relevant case of a square membrane $L_\perp \sim L_y \sim L \to \infty$, for which $L_y >> L_\perp^z$, bulk modes dominate, and we obtain,

$$R_G \propto L_\perp^\nu \ . \quad (5.41)$$

The simulations of BFT[2] measured R_G for phantom tubules by calculating the largest moment of inertia for for a set of membrane points that all had the same value of the intrinsic coordinate y. While we have used here a slightly different definition, Eq.5.25, the square root of this moment of inertia should scale like our R_G. And, indeed, BFT found that it did scale like a power of L, as in Eq.5.41, with $\nu = 0.24 \pm 0.02$ in excellent quantitative agreement with our predictions of $\nu = 1/4$, Eq.5.38 evaluated in $D = 2$. It would be of great interest to test our full anisotropic scaling prediction of Eq.5.37 by varying the aspect ratio of the membrane in such simulations. For instance, one could fix $L_\perp$ and increase L_y; we predict that one should observe *no* change in R_G. The same

should hold if one *decreased* L_y at fixed $L_\perp$: R_G should remain unchanged until $L_y \sim \sqrt{L_\perp}$, at which point the tubule should begin to get thinner (i.e. R_G should decrease).

Equations 5.36 and 5.40 also correctly recover the limit of $L_y = \text{constant} << L_\perp^z \to \infty$, where the $q_y = 0$ "zero modes" dominate, the tubule simply becomes a phantom, coiled up, $D - 1$-dimensional polymeric network of size $L_\perp$ embedded in $d-1$ dimensions, with the radius of gyration $R_G(L_\perp) \sim L_\perp^{(3-D)/2}$. In the physical dimensions ($D = 2$ and $d = 3$) in particular this gives a coiled up ideal polymer of length $L_\perp$ with $R_G \sim L_\perp^{1/2}$, as expected.

We now turn our attention to the calculation of the tubule roughness h_{rms}. As we will see, here the $\mathbf{q}_\perp = 0$ zero mode will play an essential role and will dominate the transverse undulations for "very long" tubules, which (because of anisotropic scaling) in particular includes tubules made from square membranes. Using the definition of h_{rms}, Eq.5.26, we have

$$h_{rms}^2 = 2(d-D)\left[\frac{k_B T}{L_\perp^{D-1}} \int_{L_y^{-1}} \frac{dq_y}{(2\pi)} \frac{1}{\kappa(q_y)q_y^4}(1-e^{iq_y L_y}) \right.$$
$$\left. + k_B T \int_{L_\perp^{-1}, L_y^{-1}} \frac{d^{D-1}q_\perp dq_y}{(2\pi)^D} \frac{(1-e^{iq_y L_y})}{tq_\perp^2 + \kappa q_y^4}\right] . \quad (5.42)$$

Here we observe that h_{rms} in Eq.5.42 does not receive any contribution from the $q_y = 0$ "zero mode" (i.e., in addition to the bulk mode, h_{rms} receives a contribution only from the $\mathbf{q}_\perp = 0$ "zero mode"). This is to be contrasted with the behavior of R_G that we noted following Eq.5.32, and is responsible for the differences in scaling properties of R_G and h_{rms}, notes above.

Now, for perverse and twisted reasons of our own, we choose to scale L_y, rather than $L_\perp$, out of the integrals in this expression, via the change of variables $Q_y \equiv q_y L_y, \mathbf{Q}_\perp \equiv \mathbf{q}_\perp L_y^2$, which leads to

$$h_{rms}^2 = (\frac{C_2 L_y^3}{L_\perp^{D-1}} + L_y^{5-2D} I_h(\frac{L_y}{\sqrt{L_\perp}})) \quad (5.43)$$

where

$$C_2 \equiv 2(d-D)k_B T \int_1 \frac{dQ_y}{2\pi} \frac{(1-e^{iQ_y})}{\kappa Q_y^4}. \quad (5.44)$$

is yet another constant of $O(1)$, and

$$I_h(x) \equiv 2(d-D)k_B T \int_{x^2,1} \frac{d^{D-1}Q_\perp dQ_y}{(2\pi)^D} \frac{(1-e^{iQ_y})}{tQ_\perp^2 + \kappa Q_y^4} \ .$$
$$(5.45)$$

Defining the scaling function

$$S_h(x) \equiv \sqrt{C_2 x^{2(D-1)} + I_h(x)}, \quad (5.46)$$

we see that h_{rms} can be rewritten in the scaling form

13

$$h_{rms}(L_\perp, L_y) = L_y^\zeta S_h(L_y/L_\perp^z) \qquad (5.47)$$

with, for phantom membranes,

$$\zeta = \frac{5 - 2D}{2} \ , \qquad (5.48)$$

$$z = \frac{1}{2} \ . \qquad (5.49)$$

Again, this scaling law Eq.5.47 continues to apply when self-avoidance is included, but with different values of ζ and z.

Equations 5.37 and 5.47 give information about the tubule roughness for arbitrarily large size $L_\perp$ and L_y, and arbitrary aspect ratio. For the physically relevant case of a square membrane $L_\perp \sim L_y \sim L \to \infty$, for which $L_y >> L_\perp^z$, we obtain,

$$h_{rms} \propto \frac{L_y^{\zeta + (D-1)/2z}}{L_\perp^{(D-1)/2}} \ , \qquad (5.50)$$

$$\propto L^{\zeta + (D-1)(1-z)/2z} \ , \qquad (5.51)$$

Equations 5.48, 5.49 then give, for a $D = 2$ phantom tubule, $\zeta = 1/2$, $z = 1/2$

$$h_{rms} \sim \frac{L_y^{3/2}}{L_\perp^{1/2}} \ , \qquad (5.52)$$

and therefore predicts for a square membrane

$$h_{rms} \sim L \ . \qquad (5.53)$$

This prediction for square phantom membranes has also been spectacularly quantitatively confirmed in simulations by BFT[2]. Their ingenious procedure for determining h_{rms} is rather involved, and the interested reader is referred to their paper for a clear and complete discussion of it. The bottom line, however, is that they find $h_{rms} \sim L^\gamma$, (our γ is ζ in their notation) with $\gamma = 0.895 \pm 0.06$, in excellent agreement with our prediction $\gamma = 1$ from Eq.5.53 above. As with R_G, it would be interesting to test the full scaling law Eq.5.47 by simulating non-square membranes, and testing for the independent scaling of h_{rms} with L_y and $L_\perp$. Note that, unlike R_G, according to Eq.5.52, h_{rms} will show immediate growth (reduction) when one increases (decreases) L_y at fixed $L_\perp$.

Because, unlike the flat phase, no $\log(L/a)$ correction arises, the ($D = 2$) phantom tubule is just marginally *stable*, but with wild transverse undulations which scale linearly with its length. As we will see in Sec.VI, these wild fluctuations will be suppressed when the effects of self-avoidance are included.

The above discussion also reveals that our earlier conclusions about the lower critical dimension D_{lc} for the existence of the tubule are strongly dependent on how $L_\perp$ and L_y go to infinity relative to each other; i.e., on the membrane aspect ratio. The earlier conclusion that

$D_{lc} = 3/2$ only strictly applies when the bulk modes dominate the physics, which is the case for a very squat membrane, with $L_y \approx L_\perp^z$, in which case $L_y << L_\perp$. For the physically more relevant case of a square *phantom* membrane, from the discussion above, we find that $D_{lc} = 2^-$, where the $^-$ superscript means that there are no logarithmic corrections at $D = 2$ and therefore strictly speaking the $D = 2$ tubule *is* marginally stable.

Equations 5.37 and 5.47 also correctly recover the limit of $L_\perp^z = \text{constant} << L_y \to \infty$, where the tubule simply becomes a polymer of thickness $R_G(L_\perp)$ given in Eq.5.25 of length L_y embedded in $d - 1$ dimensions. As already discussed in the Introduction for a more general case of a self-avoiding tubule, these equations then correctly recover this polymer limit giving

$$h_{rms} \approx L_P(L_y/L_P)^{3/2} \ , \qquad (5.54)$$

with $L_\perp$-dependent persistent length

$$L_P(L_\perp) \propto L_\perp^{D-1} \ . \qquad (5.55)$$

which agrees with Eq.1.10 of Sec.I, for $D = 2$ when one remembers that, for the phantom membranes, $\eta_\kappa = 0$. So, as expected for a phantom tubule, if $L_\perp$ does not grow fast enough (e.g. remains constant), while $L_y \to \infty$, the tubule behaves as a linear polymer and crumples along its axis and the distinction between the crumpled and tubule phases disappears.

To summarize: the radius of gyration R_G and the tubule roughness h_{rms} scale differently with membrane size L for a square membrane because the former is dominated by bulk modes, while the latter is dominated by $\mathbf{q}_\perp = 0$ "zero modes".

VI. SELF-AVOIDANCE IN THE TUBULE PHASE

We now look at the effects of self-avoidance on the tubule phase, and begin by calculating the upper critical embedding dimension d_{uc} below which the self-avoidance becomes relevant in the tubule phase. A model of a self-avoiding membrane in the the tubule phase is described by a free energy functional which is a combination of the elastic free energy F_{el} from Eq.5.3 and the self-avoiding interaction F_{SA} from Eq.2.1 specialized to the tubule extended in y-direction using Eq.5.1 for $\vec{r}(\mathbf{x})$

$$F_{SA} = \frac{b}{2} \int dy dy' \, d^{D-1}x_\perp d^{D-1}x'_\perp \, \delta^{(d-1)}\big(\vec{h}(\mathbf{x}_\perp, y) - \vec{h}(\mathbf{x}'_\perp, y)\big)$$
$$\times \, \delta\big(\zeta_y y + u(\mathbf{x}_\perp, y) - \zeta_y y' - u(\mathbf{x}'_\perp, y')\big) \ . \qquad (6.1)$$

If the in-plane fluctuations u scale sub-linearly with y (which we will self-consistently verify a posteriori that they do), at long length scales one can ignore the phonons inside the self-avoiding interaction above. This can be confirmed more formally by an explicit renormalization

group analysis[33]. We then obtain a self-avoiding interaction that is local in y, with corrections that are irrelevant in the renormalization group sense and therefore subdominant at long length scales. The appropriate free energy that describes a self-avoiding tubule is then given by

$$
\begin{aligned}
F = \frac{1}{2} \int d^{D-1}x_\perp dy & \Big[\gamma \Big(\partial_y u + \frac{1}{2}(\partial_y \vec{h})^2 + \frac{1}{2}(\partial_y u)^2 \Big) \\
& + \kappa(\partial_y^2 \vec{h})^2 + t(\partial_\alpha^\perp \vec{h})^2 + g_\perp(\partial_\alpha^\perp u)^2 \\
& + g_y \Big(\partial_y u + \frac{1}{2}(\partial_y \vec{h})^2 + \frac{1}{2}(\partial_y u)^2 \Big)^2 \Big] \\
& + v \int dy\, d^{D-1}x_\perp d^{D-1}x'_\perp\, \delta^{(d-1)}\big(\vec{h}(\mathbf{x}_\perp, y) - \vec{h}(\mathbf{x}'_\perp, y)\big) ,
\end{aligned}
$$
(6.2)

where $v = b/2\zeta_y$.

It is important for simulators to note that, although the self-avoiding interaction is effectively local in *intrinsic* coordinate y, this does *not* mean that the effects of self-avoidance can be included in simulations that have each particle on the membrane avoid only those labeled by the same *intrinsic* y coordinate. Such a simulation, rather, models the very different (unphysical) self-avoiding interaction

$$
\begin{aligned}
F_{SA}^{wrong} = v \int dy\, d^{D-1}x_\perp d^{D-1}x'_\perp\, \delta(u(\mathbf{x}_\perp, y) - u(\mathbf{x}'_\perp, y) \\
\times\, \delta^{(d-1)}\big(\vec{h}(\mathbf{x}_\perp, y) - \vec{h}(\mathbf{x}'_\perp, y)\big) ,
\end{aligned}
$$
(6.3)

which accounts for interaction only of particles that have the same intrinsic coordinate y *and* the same extrinsic coordinate. For large membranes, this unphysical interaction is smaller than the true self-avoiding interaction in Eq.6.2 by a factor that scales like the inverse of the rms fluctuations of u: $\langle u^2 \rangle^{-1/2}$, as can be seen trivially from the scaling of the δ-function of u in Eq.6.3. Since these fluctuations of u diverge as $L_\perp \to \infty$ like $u_{rms} \sim L_\perp^{\zeta_u}$, with $\zeta_u > 0$ (e.g., $\zeta_u = 1/8$, for $d \geq 11$ and $D = 2$), the *wrong* self-avoiding interaction in Eq.6.3 drastically *underestimates* the true self-avoiding interaction by a factor that diverges in the thermodynamic limit. Although it is tempting to do so in simulations, one must be careful not to implement the unphysical self-avoiding interaction in Eq.6.3. Since it might be difficult to implement the approximate (but asymptotically exact) self-avoiding interaction of Eq.6.2 in simulations, it is easiest to simulate the unapproximated interaction in Eq.6.1.

In the next three subsections we analyze the properties of a self-avoiding tubule described by this nonlinear elastic free energy, using Flory theory[1], the renormalization group, and the Gaussian variational method[1].

A. Flory theory

The effects of self-avoidance in the tubule phase can be estimated by generalizing standard Flory arguments from polymer physics[14] to the extended tubule geometry. The total self-avoidance energy scales as

$$
E_{SA} \propto V \rho^2 ,
$$
(6.4)

where

$$
V \propto R_G^{d-1} L_y
$$
(6.5)

is the volume in the embedding space occupied by the tubule and $\rho = M/V$ is the embedding space density of the tubule. Using the fact that the tubule mass M scales like $L_\perp^{D-1} L_y$, we see that

$$
E_{SA} \propto \frac{L_y L_\perp^{2(D-1)}}{R_G^{d-1}} ,
$$
(6.6)

Using the radius of gyration $R_G \propto L_\perp^\nu$, and considering, as required by the anisotropic scaling, a membrane with $L_\perp \propto L_y^2$, we find that $E_{SA} \propto L_y^{\lambda_{SA}}$ around the phantom fixed point, with

$$
\lambda_{SA} = 1 + 4(D - 1) - 2(d - 1)\nu ,
$$
(6.7)

Self-avoidance is relevant when $\lambda_{SA} > 0$, which, from the above equation, happens for $\nu = \nu_{ph} = (5 - 2D)/4$ (as per Eq.5.38) when the embedding dimension

$$
d < d_{uc}^{SA} = \frac{6D - 1}{5 - 2D} .
$$
(6.8)

For $D = 2$-dimensional membranes, $d_{uc}^{SA} = 11$. Thus, self-avoidance is strongly relevant for the tubule phase in $d = 3$, in contrast to the flat phase.

We can estimate the effect of the self-avoidance interactions on $R_G(L_\perp)$ in Flory theory, by balancing the estimate Eq.6.6 for the self-avoidance energy with a similar estimate for the elastic energy:

$$
E_{elastic} = t \left(\frac{R_G}{L_\perp} \right)^2 L_\perp^{D-1} L_y .
$$
(6.9)

Equating $E_{elastic}$ with E_{SA}, we obtain a Flory estimate for the radius of gyration R_G:

$$
R_G(L_\perp) \propto L_\perp^{\nu_F} , \quad \nu_F = \frac{D + 1}{d + 1} ,
$$
(6.10)

which should be contrasted with the Flory estimate of $\nu_F^c = (D + 2)/(d + 2)$ for the *crumpled* phase. The similarity of the expressions is not surprising, since for the tubule phase the y-dimension decouples in both the intrinsic and the embedding spaces and is not affected by the self-avoidance. For the physical case $D = 2$, $d = 3$ Eq.6.10 gives

$$
R_G \propto L_\perp^{3/4} ,
$$
(6.11)

15

a result that is known to be *exact* for the radius of gyration of a $D = 1$-polymer embedded in $d = 2$-dimensions.[35] Since the cross-section of the $D = 2$-tubule, crudely speaking, traces out a crumpled polymer embedded in two dimensions (see Fig.2), it is intriguing to conjecture that $\nu = 3/4$ is also the *exact* result for the scaling of the thickness of the tubule. Unfortunately, we have no strong arguments supporting this appealing conjecture.

For a square membrane, $L_y \sim L_\perp$, it is straightforward to argue, as we did previously, that the $q_y = 0$ zero modes do not contribute to R_G, and $L_\perp$ is the relevant cutoff. Hence Eq.6.10 gives the correct radius of gyration. More generally, we expect

$$R_G(L_\perp, L_y) \propto L_\perp^\nu S_R\left(\frac{L_y}{L_\perp^z}\right) , \qquad (6.12)$$

where $S_R(x)$ is the scaling function given in Eq.1.5 and z is the anisotropy exponent given in Eq.1.3.

B. Renormalization group and scaling relations

In this subsection, we present a renormalization group analysis of the physical *self-avoiding* membrane, which will also require a simultaneous treatment of the non-linear elasticity that was already present in a phantom membrane, as discussed in Sec.V.

The correct model, which incorporates the effects of both the self-avoiding interaction and the anharmonic elasticity, is defined by the free energy Eq.6.2.

$$F = \frac{1}{2} \int d^{D-1}x_\perp dy \left[\kappa(\partial_y^2 \vec{h})^2 + t(\partial_\alpha^\perp \vec{h})^2 + g_\perp(\partial_\alpha^\perp u)^2 \right.$$

$$+ g_y \left(\partial_y u + \frac{1}{2}(\partial_y \vec{h})^2 \right)^2 \right]$$

$$+ v \int dy \, d^{D-1}x_\perp d^{D-1}x_\perp' \, \delta^{(d-1)}\left(\vec{h}(x_\perp, y) - \vec{h}(x_\perp', y) \right) , \qquad (6.13)$$

where we have set $\gamma = 0$ and dropped the subdominant phonon anharmonicity.

It is convenient for the purposes of this section to choose the units of length such that $t = \kappa = 1$ throughout, and choose the renormalization group rescalings to keep them fixed at 1 even after the diagrammatic corrections are taken into account (i.e., beyond the tree-level). We follow the standard renormalization group procedure[36]:

(i) Integrate out fluctuations of the Fourier modes $u(\mathbf{q})$ and $\vec{h}(\mathbf{q})$ of the fields $u(\mathbf{x})$ and $\vec{h}(\mathbf{x})$ with wavevectors in the high wavevector shell $\Lambda e^{-l} < q_\perp < \Lambda$, $-\infty < q_y < \infty$, where the ultraviolet cutoff Λ is of order an inverse microscopic length, and l is a parameter known as the "renormalization group time". This integration can, of course only be accomplished perturbatively in the non-linear couplings v and g_y.

(ii) Anisotropically rescale lengths $(\mathbf{x}_\perp, y)$ and fields $(\vec{h}(\mathbf{x}), u(\mathbf{x}))$, so as to restore the ultraviolet cutoff to Λ:

$$\mathbf{x}_\perp = e^l \mathbf{x}_\perp' , \qquad (6.14a)$$

$$y = e^{zl} y' , \qquad (6.14b)$$

$$\vec{h}(\mathbf{x}) = e^{\nu l} \vec{h}'(\mathbf{x}') , \qquad (6.14c)$$

$$u(\mathbf{x}) = e^{(2\nu - z)l} u'(\mathbf{x}') , \qquad (6.14d)$$

where we have chosen the convenient (but not necessary) rescaling of the phonon field u so as to preserve the form of the rotation-invariant operator $(\partial_y u + \frac{1}{2}(\partial_y \vec{h})^2)^2$.

(iii) Define the effective length-scale dependent coupling constants so as to bring the resulting long wavelength effective free energy into the same form as Eq.6.13.

As discussed above, we will choose the arbitrary rescaling exponents ν and z so as to keep the renormalized $\kappa(l)$ and $t(l)$ equal to one. This choice of ν and z can be shown by standard renormalization group arguments to be the ν and z that appear in the scaling function Eqs.1.1 and 1.2, as we will demonstrate later in this subsection.

The result of the three steps of the above renormalization group transformation (i.e., mode integration, rescaling, and coupling redefinition) can be summarized in differential recursion relations for the flowing coupling constants:

$$\frac{dt}{dl} = [2\nu + z + D - 3 - f_t(v)]t , \qquad (6.15)$$

$$\frac{d\kappa}{dl} = [2\nu - 3z + D - 1 + f_\kappa(g_y, g_\perp)]\kappa , \qquad (6.16)$$

$$\frac{dg_y}{dl} = [4\nu - 3z + D - 1 - f_g(g_y)]g_y , \qquad (6.17)$$

$$\frac{dg_\perp}{dl} = [4\nu - z + D - 3]g_\perp , \qquad (6.18)$$

$$\frac{dv}{dl} = [2D - 2 + z - (d-1)\nu - f_v(v)]v , \qquad (6.19)$$

where the various f-functions represent the graphical (i.e., perturbative) corrections. Since the self-avoiding interaction only involves $\vec{h}$, and the parameters in the $\vec{h}$ propagator (t and κ) are going to be held fixed at 1, the graphical corrections coming from self-avoiding interaction alone depend only on the strength v of the self-avoiding interaction. Therefore, to *all* orders in v, and leading order in g_y, $f_t(v)$ and $f_v(v)$ are only functions of v and $f_\kappa(g_y, g_\perp)$ and $f_g(g_y)$ are only functions of g_y and $g_\perp$.

It is important to note that $g_\perp$ suffers no graphical corrections, i.e., Eq.6.18 is *exact*. This is enforced by an exact symmetry

$$u(\mathbf{x}_\perp, y) \to u(\mathbf{x}_\perp, y) + \chi(\mathbf{x}_\perp) , \qquad (6.20)$$

where $\chi(\mathbf{x}_\perp)$ is an arbitrary function of $\mathbf{x}_\perp$, under which the nonlinearities in F are invariant.

We further note that there is an additional tubule "gauge"-like symmetry for $g_y = 0$

$$\vec{h}(\mathbf{x}_\perp, y) \to \vec{h}(\mathbf{x}_\perp, y) + \vec{\phi}(y) , \qquad (6.21)$$

under which the only remaining nonlinearity, the self-avoiding interaction, being local in y, is invariant. This "tubule gauge" symmetry demands that $f_\kappa(g_y = 0, g_\perp) = 0$, which implies that if $g_y = 0$, there is no divergent renormalization of κ, $exactly$, i.e., the self-avoiding interaction $alone$ cannot renormalize κ. This non-renormalization of κ by the self-avoiding interaction, in a truncated (unphysical) membrane model with $g_y = 0$, has been recently verified to all orders in a perturbative renormalization group calculation[3].

To see that the ν and z obtained as fixed point solutions of Eqs.6.15-6.19 have the same physical significance as the ν and z defined in the scaling expressions Eqs.1.1 and 1.2 for the radius of gyration R_G and tubule wigglyness h_{rms}, we use the renormalization group transformation to relate these quantities in the unrenormalized system to those in the renormalized one. This gives, for instance, for the radius of gyration

$$R_G(L_\perp, L_y; t(0), \kappa(0), \dots) =$$
$$= \langle |\vec{h}(\mathbf{L}_\perp, y) - \vec{h}(\mathbf{0}_\perp, y)|^2 \rangle^{1/2} \Big|_{L_y, t(0), \kappa(0), \dots}$$
$$= e^{\nu l} \langle |\vec{h}(e^{-l}\mathbf{L}_\perp, y) - \vec{h}(\mathbf{0}_\perp, y)|^2 \rangle^{1/2} \Big|_{e^{-zl}L_y, t(l), \kappa(l), \dots}$$
$$= e^{\nu l} R_G(e^{-l}L_\perp, e^{-zl}L_y; t(l), \kappa(l), \dots) ,$$
$$\qquad (6.22)$$

where $t(l), \kappa(l), \dots$ stand for all flowing coupling constants whose evolution with l is determined by the recursion relations Eqs.6.15-6.19. Choosing $l = l_* = \log L_\perp$ this becomes:

$$R_G(L_\perp, L_y; t, \kappa, \dots) = L_\perp^\nu R_G(1, L_y/L_\perp^z; t(l_*), \kappa(l_*), \dots) .$$
$$\qquad (6.23)$$

This relation holds for any choice of the (after all, arbitrary) rescaling exponents ν and z. However, if we make the special choice such that Eqs.6.15-6.19 lead to fixed points (see Eqs.6.36-6.39), $t(l_*)$, $\kappa(l_*)$, ... in Eq.6.23 go to $constants$, independent of l_* (and hence $L_\perp$), as $L_\perp$ and hence l_*, go to infinity. Thus, in this limit, we obtain from Eq.6.23

$$R_G(L_\perp, L_y; t, \kappa, \dots) = L_\perp^\nu R_G(1, L_y/L_\perp^z; t_*, \kappa_*, \dots) ,$$
$$\qquad (6.24)$$

where $t_*, \kappa_*, \dots$ are the fixed point values of coupling constants. This result clearly agrees with the scaling forms for R_G, Eq.1.1 (with analogous derivation for h_{rms}) if we define $S_R(x) \equiv R_G(1, x; t_*, \kappa_*, g_y^*, v^*)$.

The recursion relations Eqs.6.15-6.19 reproduce all of our phantom membrane results, as well as the upper critical embedding dimension d_{uc}^{SA} for self-avoidance predicted by Flory theory, Eq.6.7, and the upper critical

$intrinsic$ dimension $D_{uc} = 5/2$ for anomalous elasticity for phantom membranes. To see this, consider first the phantom membrane; i.e., $v = 0$. In this case, $f_t(v) = 0$, and to keep $t(l)$ fixed we see from the recursion relation Eq.6.15 for $t(l)$ that we must choose

$$2\nu + z + D - 3 = 0 . \qquad (6.25)$$

Assuming for the moment that $f_\kappa(g_y, g_\perp) \to 0$ as $l \to \infty$, which, as we shall see in a moment, it does for phantom membranes for $D > 3/2$, we see from the recursion relation Eq.6.16 for $\kappa(l)$ that we must choose

$$2\nu - 3z + D - 1 = 0 . \qquad (6.26)$$

Solving Eqs.6.25 and 6.26 for z and ν yields the phantom membrane results $z = 1/2$, $\nu = (5 - 2D)/4$, as obtained in Eqs.5.38 and 5.39.

To extract the upper-critical embedding dimension d_{uc}^{SA} for self-avoidance from the renormalization group recursion relations, we construct from them a flow equation for a dimensionless coupling constant

$$\tilde{v} = v t^a \kappa^b , \qquad (6.27)$$

where a and b will be chosen to eliminate the arbitrary rescaling exponents ν and z from the recursion relation for $\tilde{v}$. This requirement lead to the choice

$$a = (3d - 5)/8 , \qquad (6.28)$$
$$b = (d + 1)/8 , \qquad (6.29)$$

which implies:

$$\frac{d\tilde{v}}{dl} = \left([6D - 1 - (5 - 2D)d]/4 - f_v + \frac{d+1}{8} f_\kappa \right.$$
$$\left. - \frac{3d-5}{8} f_t \right) \tilde{v} , \qquad (6.30)$$

Of course, an identical flow equation is obtained for $v(l)$ if one instead requires that $t(l)$ and $\kappa(l)$ are fixed, i.e., independent of l, thereby determining ν and z and using them inside Eq.6.19.

It is easy to see that the sign of the terms in the square bracket determines the relevance of the self-avoiding interaction, which becomes relevant when

$$6D - 1 - (5 - 2D)d > 0 , \qquad (6.31)$$

i.e., for $d < d_{uc}^{SA} = (6D - 1)/(5 - 2D)$, consistent with the analysis of the Flory theory, Eq.6.7.

Likewise, the renormalization group flow equations contain information about the upper-critical $intrinsic$ dimension for the anomalous elasticity, D_{uc}, below which tubule elasticity becomes anomalous. This can be seen (analogously to the discussion of the relevance of self-avoidance coupling v) by using Eqs.6.15-6.17 to construct the renormalization group flow equation for the dimensionless coupling constant

$$\tilde{g}_y = \frac{g_y}{t^{3/4}\kappa^{5/4}} \;, \qquad (6.32)$$

chosen such that its flow

$$\frac{d\tilde{g}_y}{dl} = \left(\frac{5}{2} - D - f_g - \frac{5}{4}f_\kappa + \frac{3}{4}f_t\right)\tilde{g}_y \;, \qquad (6.33)$$

is independent of the arbitrary rescaling exponents z and ν. Again the same recursion relation can be obtained by instead using the values of z and ν required to keep $t(l)$ and $\kappa(l)$ fixed inside the flow equation for $g_y(l)$, Eq.6.17. It is then obvious that anharmonic elasticity becomes relevant for $D < D_{uc} = 5/2$, where anomalous elasticity of the tubule is induced. As we will see below, in a phantom tubule or a tubule embedded in $d > d*$, this anomalous elasticity manifests itself *only* in phonon (u) fluctuations, i.e., softens g_y, but does *not* renormalize the bending rigidity κ. In *physical* tubules, however, which are self-avoiding and are embedded in $d = 3 < d_* \approx 6.5$, the elasticity is fully anomalous, both with respect to the phonon u fluctuations (i.e. g_y vanishes as $q \to 0$) and the height $\vec{h}$ undulations (i.e. κ diverges as $q \to 0$).

To further analyze the renormalization of κ in a self-avoiding membrane, it is convenient to integrate out the phonon field u as we did in Sec.V for the phantom tubule, obtaining

$$F = \frac{1}{2}\int d^{D-1}x_\perp dy \left[\kappa(\partial_y^2\vec{h})^2 + t(\partial_\alpha^\perp\vec{h})^2\right]$$
$$+ F_{anh}[\vec{h}] + F_{SA}[\vec{h}] \qquad (6.34)$$

where, F_{anh} is the non-local interaction, Eq.5.11, mediated by integrated out phonons, with a kernel

$$V_h(\mathbf{q}) = \frac{g_y g_\perp q_\perp^2}{g_y q_y^2 + g_\perp q_\perp^2} \;, \qquad (6.35)$$

and F_{SA} is the self-avoiding interaction.

The long wavelength properties of the tubule phase will very much depend on the behavior of the denominator in the kernel V_h at long length scales. If $g_y(\mathbf{q})q_y^2 >> g_\perp(\mathbf{q})q_\perp^2$ (as we saw for a phantom tubule) then at long scales $V_h(\mathbf{q}) \approx g_\perp q_\perp^2/q_y^2$, which behaves like $\sim q_y^2$ in the relevant limit of $q_\perp \sim q_y^2$. In this case, simple power counting around the *Gaussian* fixed point then shows that this elastic nonlinearity only becomes relevant for $D < D_{uc} = 3/2$, i.e. is irrelevant for a physical $D = 2$-dimensional tubule, as we argued in Sec.V.

On the other hand, if the scaling is such that $g_\perp(\mathbf{q})q_\perp^2$ dominates over $g_y(\mathbf{q})q_y^2$, then $V_h(\mathbf{q}) \approx g_y$, i.e. a constant at long length scales. Simple power-counting then shows that this coupling is relevant for $D < D_{uc} = 5/2$ and the bending rigidity modulus of a $D = 2$-dimensional tubule *is* anomalous in this case.

As we saw in our analysis of a *phantom* tubule, for which one is perturbing around a *Gaussian* fixed point described by $q_\perp \sim q_y^2 << q_y$ (in the long wavelength limit), the anharmonic nonlinearity is irrelevant for $D >$

3/2 and κ is *not* anomalous. We now need to extend this analysis to a physical tubule, i.e., to include the effects of self-avoidance.

The analysis of the behavior of $V_h(\mathbf{q})$ (which determines the relevance of anharmonic elasticity) at long scales, around an arbitrary fixed point, is more conveniently done using the language of the renormalization group through the recursion relations Eqs.6.15 and 6.19. At the globally stable fixed point, in the presence of both the nonlinear elasticity and the self-avoiding interaction, we can keep $t = \kappa = 1$ and g_y and v fixed at fixed point values, by requiring

$$2\nu + z + D - 3 - f_t(v^*) = 0 \;, \qquad (6.36)$$
$$2\nu - 3z + D - 1 + f_\kappa(g_y^*, g_\perp^*) = 0 \;, \qquad (6.37)$$
$$4\nu - 3z + D - 1 - f_g(g_y^*) = 0 \;, \qquad (6.38)$$
$$2(D-1) + z - \nu(d-1) - f_v(v^*) = 0 \;. \qquad (6.39)$$

In light of the above discussion, the anharmonic vertex for $\vec{h}$ in this renormalization group picture becomes relevant when $g_\perp(l \to \infty)$ renormalizes to infinity, while it is *irrelevant* when $g_\perp(l \to \infty)$ flows to zero. Thus, the relevance of V_h is decided by the sign of the renormalization group flow eigenvalue of $g_\perp(l)$ in Eq.6.18

$$\lambda_{g_\perp} = 4\nu - z + D - 3 \;, \qquad (6.40)$$

which is *exactly* determined by the values of ν and z, since $g_\perp$ suffers no graphical renormalization.

As we have discussed in previous sections, for a phantom tubule $\nu = (5 - 2D)/4$ and $z = 1/2$. For $d < d_{uc}^{SA} = (6D-1)/(5-2D) (= 11$ for $D = 2)$, these values are modified by the self-avoiding interaction, but only by order $\epsilon \equiv d - d_{uc}^{SA}$, i.e.

$$\nu = (5 - 2D)/4 + O(\epsilon) \;, \qquad (6.41)$$
$$z = 1/2 + O(\epsilon) \;. \qquad (6.42)$$

Hence a $D = 2$-dimensional tubule, embedded in d dimensions close to $d_{uc}^{SA} = 11$, $\lambda_{g_\perp} = -1/2$ and $g_\perp(l)$ flows according to

$$\frac{dg_\perp}{dl} = \left[-\frac{1}{2} + O(\epsilon)\right]g_\perp \;, \qquad (6.43)$$

i.e. $g_\perp$ is *irrelevant* near $d = 11$ (for $\epsilon \ll 1$), $V_h(\mathbf{q}) \sim g_\perp q_\perp^2/q_y^2 \sim q_y^{2-O(\epsilon)}$ is irrelevant for a physical $D = 2$-dimensional tubule , and, hence, f_κ in Eq.6.16 vanishes as $l \to \infty$. So κ is unrenormalized near $d = 11$, for $D = 2$. That is, as we described above, the anharmonic elasticity is irrelevant to the *bend* elasticity for embedding dimensions near d_{uc}^{SA}, and in this case the full model of a self-avoiding tubule with nonlinear elasticity reduces to the *linear* elastic truncated model introduced by us[1] and recently further analyzed in Ref. 3.

In this simpler (but unphysical) case, one is justified in ignoring the nonlinear elasticity. One is then able to analyze (perturbatively in $\epsilon = d_{uc}^{SA} - d$) the effects of the

18

self-avoiding interaction alone, by computing the functions $f_t(v)$ and $f_v(v)$ appearing in Eqs.6.15 and 6.19.[3] Since, as we discussed above, the "tubule gauge" symmetry guarantees that in this case the self-avoiding interaction alone cannot renormalize κ, $f_\kappa = 0$. Thus, for d near d_{uc}^{SA}, Eq.6.37, leads to $\eta_\kappa = 0$ and an *exact* exponent relation (leaving only a single independent tubule shape exponent):

$$z = \frac{1}{3}(2\nu + D - 1) , \qquad (6.44)$$

which is exact for a finite *range* $d_* < d < d_{uc}^{SA}$ of embedding dimensions, and for phantom tubules in any embedding dimension. This result has been independently obtained in Ref. 3.

However, this simple scenario, and, in particular, the scaling relation Eq.6.44, is *guaranteed* to break down as d is reduced. The reason for this is that, as d decreases, ν increases, and eventually becomes so large that the eigenvalue $\lambda_{g_\perp}$ of $g_\perp$ changes sign and becomes positive. As discussed earlier, once this happens, the nonlinear vertex Eq.6.35 becomes relevant, and κ acquires a divergent renormalization, i.e., $f_\kappa \neq 0$, and bend tubule elasticity becomes anomalous. We will now show that the critical dimension d_* below which this happens for $D = 2$ is *guaranteed* to be $> 7/2$, and hence, obviously, > 3.

To show this, we use the exponent relation Eq.6.44, which is valid for $d > d_*$, inside the expression for the eigenvalue $\lambda_{g_\perp}$, Eq.6.40, obtaining

$$\lambda_{g_\perp} = \frac{1}{3}(10\nu + 2D - 8) . \qquad (6.45)$$

We then take advantage of a rigorous lower bound on ν

$$\nu > \frac{D-1}{d-1} , \qquad (6.46)$$

imposed by the condition that the monomer density $\rho \propto L_\perp^{D-1}/R_G^{d-1} \propto L_\perp^{D-1-\nu(d-1)}$ remain finite in the thermodynamic $L_\perp \to \infty$ limit. Using this bound inside Eq.6.45 we obtain

$$\lambda_{g_\perp} \geq \frac{1}{3}\left(10\frac{D-1}{d-1} + 2D - 8\right) , \qquad (6.47)$$

from which it follows that $\lambda_{g_\perp}$ *must* become positive for $d < d_*^{lb}(D)$ with

$$d_*^{lb}(D) = \frac{4D-1}{4-D} , \qquad (6.48)$$

$$d_*^{lb}(2) = 7/2 , \qquad (6.49)$$

as asserted above.

In fact, $d_*(2)$ is probably quite a bit bigger than its $7/2$ lower bound, as two estimates of it indicate. If, for example, we take the Flory tubule exponent $\nu = (D+1)/(d+1)$ in Eq.6.45, we obtain:

$$d_*^F = \frac{6D+1}{4-D} , \qquad (6.50)$$

$$d_*^F(2) = 13/2 , \qquad (6.51)$$

while if we use the $\epsilon = 11 - d$–expansion result for ν of Bowick and Guitter, $(D = 2)$[3]

$$\nu_\epsilon = \frac{3}{4+\delta} - \frac{1}{2} , \qquad (6.52)$$

with

$$\delta = -1.05\left(\frac{\epsilon}{8}\right) , \qquad (6.53)$$

we obtain

$$d_*^\epsilon = 5.92 . \qquad (6.54)$$

So, based on the above estimates, we expect that in a $D = 2$-dimensional tubule, embedded in $d < d_* \approx 6$, the fixed point of the truncated tubule model introduced by us[1] and studied in Ref. 3, is *unstable* to anharmonic elasticity F_{anh}. This means that κ diverges at long length scales, and the scaling relation Eq.6.44 between z and ν breaks down. Thus, for the physical embedding dimension $d = 3$, the tubule bend elasticity is certainly anomalous, in the sense that κ diverges, and probably quite strongly. We have summarized the above discussion in Fig.6, schematically illustrating how the renormalization group flow of $g_\perp$, and therefore the anomalous κ elasticity, change (at d_*) as a function of embedding dimension d.
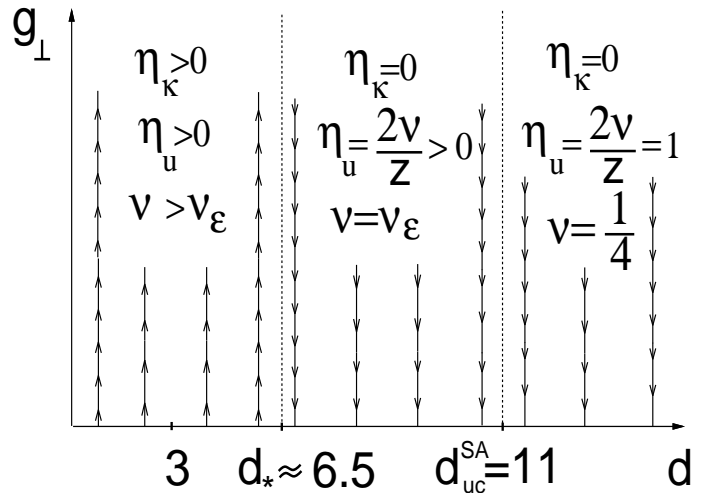

FIG. 6. Schematic illustration (specialized to $D = 2$) of change in relevance of $g_\perp(l)$ which occurs at d_*. For embedding dimensions below d_* (which includes the physical case of $d = 3$), $g_\perp(l)$ becomes relevant, leading to anomalous bending elasticity with $\kappa(\mathbf{q}) \sim q_y^{-\eta_\kappa}$, which diverges at long length scales. Other consequences of this qualitative and quantitative change for $d < d_*$ are discussed in the text.

Once $d < d_*$, the new nontrivial relations Eqs.6.37 and 6.38 hold, with functions $f_\kappa(g_y, g_\perp)$ and $f_g(g_y)$ evaluated at the fixed point values g_y^* and $g_\perp^*$.

Using the sort of renormalization group correlation function matching calculations described earlier, Eqs.6.22-6.24, it is straightforward to show that the correlation functions of the tubule, including anomalous elastic effects, are correctly given by the harmonic results, Eqs.5.6 and 5.7, *except* that the elastic *constants* g_y and κ must be replaced by wavevector dependent quantities that vanish and diverge, respectively as $\mathbf{q} \to 0$:

$$g_y(\mathbf{q}) = q_y^{\eta_u} S_g(q_y/q_\perp^z) , \qquad (6.55)$$

$$\kappa(\mathbf{q}) = q_y^{-\eta_\kappa} S_\kappa(q_y/q_\perp^z) , \qquad (6.56)$$

with

$$z\eta_\kappa = f_\kappa(g_y^*, g_\perp^*) , \qquad (6.57)$$

$$z\eta_u = f_g(g_y^*) . \qquad (6.58)$$

Our earlier conclusion that the relevance of V_h is determined by the sign of $\lambda_{g_\perp}$ (Eq.6.40) can be reproduced by simply noting that $g_y(\mathbf{q})q_y^2$ scales like $q_y^{\eta_u+2}$, and in the long wavelength limit is therefore subdominant to $g_\perp q_\perp^2 \sim q_y^{2/z}$ when

$$z\eta_u > 2 - 2z , \qquad (6.59)$$

which, upon using Eq.6.38 and the definition of $\eta_u = f_g(g_y^*)/z$, is identical to the condition that $\lambda_{g_\perp} > 0$.

The scaling functions have the asymptotic forms

$$S_g(x \to 0) \to x^{-\eta_u} , \qquad (6.60)$$

$$S_\kappa(x \to 0) \to x^{\eta_\kappa} . \qquad (6.61)$$

Combining the expressions Eqs.6.57 and 6.58 for η_κ and η_u with the RG fixed point conditions Eqs.6.37 and 6.38 shows that, at this new globally stable fixed point, *two* exact relations hold between *four* independent exponents z, ν, η_κ, and η_u (instead of a single relation Eq.6.44 between two exponents)

$$z = \frac{1}{3 - \eta_\kappa}(2\nu + D - 1) , \qquad (6.62)$$

$$z = \frac{1}{3 + \eta_u}(4\nu + D - 1) . \qquad (6.63)$$

That is, in contrast to the behavior for $d > d_*$, for $d < d_*$ there are *two* independent exponents characterizing the tubule phase, *not one*. We furthermore note that these exponent relations automatically contain the rotational symmetry Ward identity. This can be easily seen by eliminating ν from Eqs.6.62 and 6.63, obtaining

$$2\eta_\kappa + \eta_u = 3 - (D - 1)/z . \qquad (6.64)$$

Ultimately, the origin of this relation is the requirement that graphical corrections do not change the form of the rotationally invariant operator $(\partial_y u + \frac{1}{2}(\partial_y \vec{h})^2)$.

Just as the divergence of κ is controlled by $f_\kappa(g_y^*, g_\perp^*)$, the softening of $g_y(\mathbf{q}) \sim q_y^{\eta_u}$ is determined by the $\eta_u = zf_g(g_y^*)$. Because $f_g(0) = 0$, this *physical* $g_y(\mathbf{q})$ remains non-zero and finite as $q \to 0$, *only if* the running coupling $g_y(l)$ in the renormalization group recursion equation Eq.6.17 *does* go to zero (because then the graphical piece $f_g(g^*)$ vanishes). Examining the flow equation for $g_y(l)$, Eq.6.17, for $g_y(l)$ to vanish, we must have

$$4\nu - 3z + D - 1 < 0 . \qquad (6.65)$$

However, using the lower bound on ν, Eq.6.46 in the physical case of $D = 2$ and $d = 3$, we find $\nu > 1/2$. Hence, as long as $z < 1$, the condition Eq.6.65 is *not* satisfied, and therefore $g_y(\mathbf{q} \to 0) \to 0$, that is, $\eta_u > 0$. We summarize the above discussion in Fig.7.
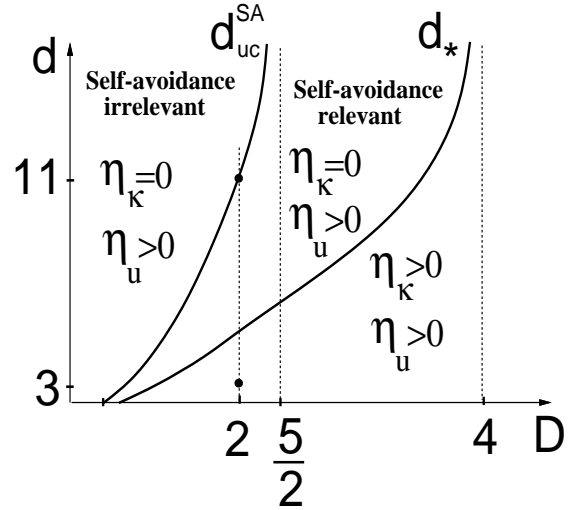

FIG. 7. Schematic of the tubule "phase" diagram in the embedding d vs intrinsic D dimensions. Self-avoiding interaction becomes relevant for $d < d_{uc}^{SA}(D) = (6D - 1)/(5 - 2D)$, ($= 11$, for $D = 2$). Below the $d_*(D)$ curve (for which the lower bound is $d_*^{lb}(D) = (4D - 1)/(4 - D)$) the anharmonic elasticity becomes relevant, leading to anomalous elasticity with a divergent bending rigidity.

We now show that the above general analysis of tubule anomalous elasticity in the presence of self avoidance, obtained using the renormalization group, can be reproduced via a heuristic, but beautiful physical argument similar to that used by Landau and Lifshitz[27] to derive shell theory. For a tubule of diameter R_G, the non-zero shear g_y elasticity leads to an effective R_G-dependent bending rigidity modulus which will be $L_\perp$ and L_y-dependent if the tubule diameter depends on $L_\perp$ and L_y. This can be seen as follows (see Fig.8):
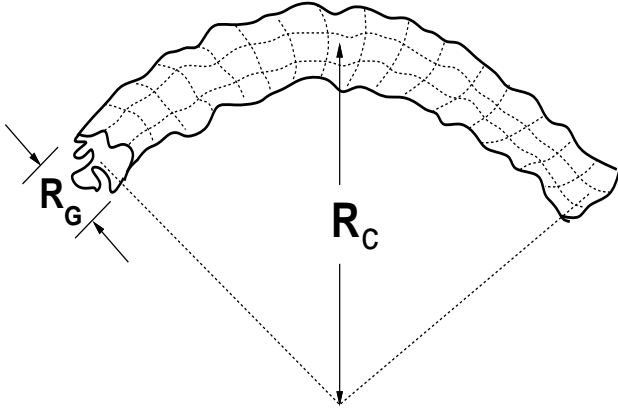
FIG. 8. Illustration of the physical mechanism for the enhancement of the bending rigidity κ by the shear g_y elasticity. To bend a polymerized tubule of thickness R_G into an arc of radius R_c requires R_G/R_c fraction of bond stretching and therefore costs elastic shear energy, which when interpreted as bending energy leads to a length-scale dependent renormalization of the bending rigidity κ and to the Ward identity Eq.6.67, as described in more detail in the text.

If we bend the tubule with some radius of curvature $R_c \gg R_G$, simple geometry tells us that this will induce a strain $\varepsilon \sim \partial_y u$ *along* the tubule axis of order $\varepsilon \sim R_G/R_c$, since the outer edge of the tubule must be stretched by this factor, and the inner edge compressed by it, in order to accomplish the required bend. This strain induces an additional elastic energy density (i.e., additional to those coming from the bare κ), namely those coming from the u elastic energy. This goes like $g_y(L_y, L_\perp)\varepsilon^2 = g_y(L_y, L_\perp)(R_G(L_y)/R_c)^2$. Interpreting this additional energy as an effective bending energy density $\kappa_y(L_\perp, L_y)/R_c^2$, leads to the *effective* bending modulus $\kappa_y(L_\perp, L_y)$,

$$\kappa_y(L_\perp, L_y) \sim g_y(L_\perp, L_y) R_G(L_\perp, L_y)^2 , \qquad (6.66)$$

Inserting the scaling forms $\kappa_y(L_\perp, L_y) = L_y^{\eta_\kappa} S_\kappa(L_y/L_\perp^z)$, $g_y(L_\perp, L_y) = L_y^{-\eta_u} S_g(L_y/L_\perp^z)$ and $R_G(L_\perp, L_y) = L_\perp^\nu S_R(L_y/L_\perp^z)$ into above expression, we obtain a relation between the scaling exponents

$$2\nu = z(\eta_\kappa + \eta_u) . \qquad (6.67)$$

which is exactly the exponent relation one obtains by subtracting Eq.6.37 from Eq.6.38, and using the expressions Eq.6.57 and 6.58 for η_κ and η_u, all of which were obtained using renormalization group arguments.

Since the above physical shell argument is very general, Eqs.6.66 and 6.67 hold independent of the mechanism that generates anomalous elasticity. For the case of the phantom membrane (for $D > 3/2$) Eq.6.66 reveals that κ is not anomalous because the softening of the shear modulus $g_y(\mathbf{q})$ by thermal fluctuations precisely compensates the bending rigidity produced by the finite diameter R_G of the tubule. Equation 6.67 then correctly predicts

for the *phantom* tubule that $\eta_u = 2\nu/z$, which is consistent with the phantom tubule results $\eta_u = 5 - 2D$, $\nu = (5 - 2D)/4$, and $z = 1/2$. Furthermore, because the anharmonic elasticity $V_h(\mathbf{q})$ is irrelevant for $d > d_*$,

$$\eta_u = 2\nu/z , \qquad (6.68)$$

is valid, even in a self-avoiding tubule embedded in these high dimensions.

We note, finally, that all of the exponents must show a jump discontinuity at d_*, as shown in Fig.9. Therefore, unfortunately, an extrapolation from $\epsilon = 11 - d$–expansion in a truncated model with linear elasticity[3] down to the physical dimension of $d = 3$ (which is below d_*) gives little information about the properties of a real tubule.
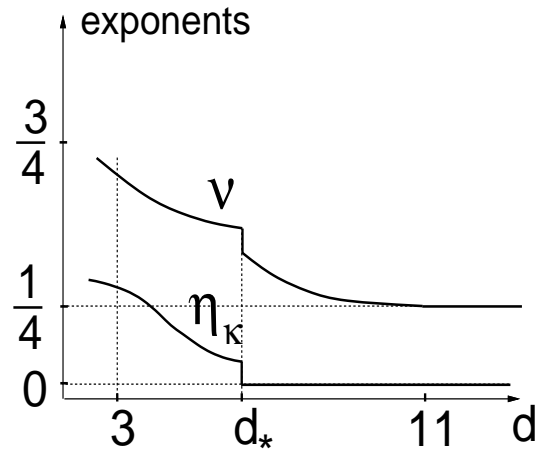

FIG. 9. Schematic graph of the shape exponent ν and anomalous bend exponent η_κ (for $D = 2$). Note the jump discontinuity as a function of embedding dimension d, occurring at $d = d_* \approx 6$.

The computations for a physical tubule must be performed for $d < d_*$, where both the self-avoidance and the anharmonic nonlinearities are relevant and must be handled simultaneously. As we discussed above, for $d < d_*$, the eigenvalue $\lambda_{g_\perp} > 0$, leading to the flow of $g_\perp(l)$ to infinity, which in turn leads to $V_h(\mathbf{q}) = g_y$. Physically this regime of $g_\perp \to \infty$ corresponds to freezing out the phonons u, i.e. setting $u = 0$ in the free energy $F[\vec{h}, u]$ in Eq.6.13. This is consistent with our finding that for $d < d_*$, in the effective free energy $F[\vec{h}]$ (with phonons integrated out), Eq.6.34, the kernel $V_h = g_y$. The resulting effective free energy functional for a physical self-avoiding tubule is

$$F = \frac{1}{2} \int d^{D-1}x_\perp dy \left[\kappa(\partial_y^2 \vec{h})^2 + t(\partial_\alpha^\perp \vec{h})^2 + \frac{1}{4}g_y(\partial_y \vec{h})^4 \right.$$
$$\left. + v \int dy \, d^{D-1}x_\perp d^{D-1}x'_\perp \, \delta^{(d-1)}\big(\vec{h}(\mathbf{x}_\perp, y) - \vec{h}(\mathbf{x}'_\perp, y)\big) \right. ,$$
$$(6.69)$$

21

Unfortunately, no controlled perturbative study is possible for $d < d_*$, since one must perturb in g_y around a nontrivial, *strong* coupling fixed described by $v^* = O(1)$ and $g_y^* = 0$. Furthermore, as we will show below, at *this* fixed point there is no upper critical dimension for g_y, i.e. anharmonic nonlinearities are always relevant for $d = 3 < d_*$, for *any* D. This strongly contrasts with the Gaussian fixed point (describing phantom membranes) at which the anharmonic nonlinearity is only relevant for $D < D_{uc} = 5/2$.

In what follows, we will illustrate how one might attempt to actually calculate the exponents ν, z, η_κ, and η_u, for $d < d_*$, and enumerate the (many) technical difficulties that prevent us from doing so, and conclude with a cautionary list of several unsuccessful uncontrolled approximations that we have tried.

In principle, all we need to do is calculate the f_i ($i = t, v, g, \kappa$) functions in the recursion relation Eqs.6.15-6.19, which represent the perturbative ("graphical") corrections to the associated coupling constants. Once these f-functions are known they give 4 equations (Eqs.6.36-6.39) that uniquely determining the 4 unknowns tubule shape exponents, ν, z, v^*, and g_y^*, as well as the the flow of $g_\perp(l)$, and therefore completely characterize the long wavelength properties of self-avoiding anharmonic tubules.

Our goal then is to calculate $f_t(v)$, $f_v(v)$, $f_\kappa(g_y)$, and $f_g(g_y)$. The functions $f_g(g_y)$ and $f_\kappa(g_y)$ are determined by the diagrammatic corrections to g_y and κ, with the corresponding Feynman diagrams displayed in Fig.10. The results to leading order in g_y, are

$$f_\kappa(g_y) = C_\kappa g_y^2 , \qquad (6.70)$$

$$f_g(g_y) = C_g g_y , \qquad (6.71)$$

where C_κ and C_g are d and D-dependent constants, whose calculation proves to be the sticking point, as we will describe below.
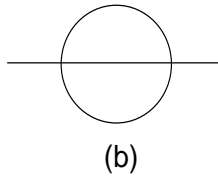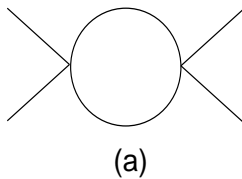
(a)

(b)

FIG. 10. Feynman graphs that renormalize: (a) the anharmonic elasticity g_y, and (b) the bending rigidity κ.

Of course, once d is below d_*, *no matter how close it is to d_**, the fixed point that controls the elastic properties of the tubule phase is *not* perturbative in g_y. That is, we do *not* expect g_y to be $O(d_* - d)$, but, rather, $O(1)$, even for $d_* - d << 1$. Furthermore, of course, since $d_* \approx 6$, $d_* - d$ is not small in the physical case $d = 3$ anyway. For both of these reasons, truncating the calculations of f_κ and f_g at the leading order in g_y, as we have done in Eqs.6.70 and 6.71, is an uncontrolled, and far from trustworthy approximation. However, we know of no other analytical approach. Furthermore, as we shall see, even this uncontrolled analytic approach proves intractable: a reliable calculation of the values of the constants C_κ and C_g has eluded us.

To complete the characterization of the fixed point we can proceed in two ways. The most direct way is to simply perturbatively evaluate the functions $f_t(v)$, $f_v(v)$. Luckily (for us) this has recently been done by Bowick and Guitter[3] in a truncated harmonic tubule model (previously introduced and studied by us[1]) near $d = d_{uc}^{SA}$. Although, for the reasons that we discussed above, these calculations are not rigorously applicable to a physical tubule in $d = 3 < d_*$ (where anharmonic elasticity is certainly important), for lack of being able to do any better we extrapolate *these* functions, computed near $d = 11$,[3] down to $d = 3$

$$f_t(v) = C_t v , \qquad (6.72)$$

$$f_v(v) = C_v v , \qquad (6.73)$$

Now using Eqs.6.70–6.73 in Eqs.6.36–6.39 we obtain four equations for four unknowns (z, ν, g_y^*, and v^*), expressed in terms constants C_κ, C_g, C_t and C_v, (specialized here to $D = 2$).

$$2\nu + z - 1 - C_t v^* = 0 , \qquad (6.74)$$

$$2\nu - 3z + 1 + C_\kappa g_y^{*2} = 0 , \qquad (6.75)$$

$$4\nu - 3z + 1 - C_g g_y^* = 0 , \qquad (6.76)$$

$$2 + z - \nu(d - 1) - C_v v^* = 0 . \qquad (6.77)$$

where the constants C_t and C_v (computed in the truncated tubule model near $d = d_{uc}^{SA}$ for $D = 2$) are given by[3]

$$C_t = \frac{1}{8\pi^2} , \qquad (6.78)$$

$$C_v = \frac{0.068}{\pi^{5/2}} , \qquad (6.79)$$

These equations can be uniquely solved for ν, z, g_y^*, and v^*. In terms of C_κ and C_g, in $D = 2$ and $d = 3$ we obtain for ν and z

$$\nu = \frac{1}{4C_\kappa} , \qquad (6.80)$$

$$z = \frac{1}{3} + \frac{1}{6C_\kappa} + \frac{C_g}{6C_\kappa} , \qquad (6.81)$$

22

from which η_κ and η_u can also be determined using the solution for g_y^* inside Eqs.6.57 and 6.58

$$\eta_\kappa = \frac{3C_g}{1 + C_g + 2C_\kappa} , \qquad (6.82)$$

$$\eta_u = \frac{3 - 3C_g}{1 + C_g + 2C_\kappa} , \qquad (6.83)$$

Another approach to estimating the tubule shape exponents is to rely on the usual accuracy of the Flory theory (in treating the effects of self-avoidance), instead of the extrapolation of functions $f_v(v)$ and $f_t(v)$ down from ϵ-expansion. Although it is usually not stated this way, in the language of renormalization group, Flory theory amounts to assuming that the graphical corrections to t and to v are the same, i.e. $f_v(v^*) = f_t(v^*)$. Using this in Eqs.6.36 and 6.39, we obtain the Flory result for ν

$$\nu_F = \frac{D+1}{d+1} , \qquad (6.84)$$

$$= \frac{3}{4} , \quad \text{for } d = 3 , \; D = 2 , \qquad (6.85)$$

consistent with our earlier analysis in subsection VI A. Note that, if $f_v(v) = f_t(v)$ for *all* v, this result would be exact *independent* of the jump in the other exponents z, η_κ, and η_u at d_*. That is, it would apply even *below* d_*, and ν would not jump, or be in any way non-analytic, at d_*.

Now, of course, we know from the explicit leading order calculation in Ref. 3 that $f_v(v)$ does *not* $= f_t(v)$ exactly. However, we *do* know from that calculation that they are quite close, at least to leading order, as illustrated by the good agreement between Flory theory and the extrapolated ϵ-expansion. *If* this persists down to $d = 3$, and to large v, and our experience with polymers suggests that it will, then ν may be quite accurately predicted by Flory theory, *despite* the complications associated with the onset of anomalous bend elasticity at d_*.

Using the Flory value for ν (Eq.6.84) inside Eqs.6.75 and 6.76, together with the diagrammatic corrections to κ and g_y given in Eqs.6.70 and 6.71, we obtain two equations (specialized to $D = 2$)

$$6/(d+1) - 3z + 1 + C_\kappa g_y^{*2} = 0 , \qquad (6.86)$$
$$12/(d+1) - 3z + 1 - C_g g_y^* = 0 , \qquad (6.87)$$

which gives for $d = 3$

$$z = \frac{4}{3} + \frac{C_g^2}{6C_\kappa} - \frac{C_g(C_g^2 + 6C_\kappa)^{1/2}}{6C_\kappa} , \qquad (6.88)$$

Now, at least in this uncontrolled approximation of truncated perturbation theory at one loop order, it seems that we are left with the straightforward task of calculating the constants C_κ and C_g. Alas, things are not so simple, for reasons that are undoubtedly connected with the fact that d_* is not perturbatively close to d_{uc}^{SA}, which

is the only dimension about which one can do a genuinely controlled approximation[1,3], *and* the much more surprising fact that, even though $\epsilon_I \equiv 5/2 - D$ is only $1/2$ (for $D = 2$), this ϵ_I-expansion in *intrinsic* dimension, as we will show, is demonstrably *extremely* unreliable, giving *qualitatively* different answers, such as a *reduction*, rather than an increase of κ due to fluctuations.

Our unsuccessful (but heroic) attempts to calculate C_g and C_κ were as follows:

(I)

Calculate them in an $\epsilon_I \equiv 5/2 - D$-expansion for a *phantom* membrane, then use these same constants C_g and C_κ for the real, self-avoiding membrane. This approach obviously makes many errors, since, by the time we get down to $d_*(5/2)$, the correlation functions of the true, self-avoiding membrane are already quite different from those of the phantom membrane, due to the effects of self-avoidance. Furthermore, these effects are particularly pronounced for intrinsic dimensions $D = 5/2$, since $d_{uc}^{SA}(5/2) = \infty$, as illustrated in Fig.7.

Nonetheless, since no other analytical calculation is available (and we are persistent young lads), we attempted this $\epsilon_I \equiv 5/2 - D$-expansion. However, the results made no physical sense: we found a *negative* η_κ, i.e., a *downward* renormalization of κ. The detailed calculations are virtually identical to those for the renormalization of κ at the tubule-to-crumpled phase transition, which are described in Sec.VII. We note here simply that the origin of this negative contribution to κ is a negative region of the real-space correlation function $G(\mathbf{x}_\perp, y) = C(x_\perp/y^2)^{1/4} Y(x_\perp/y^2)$, as given by Eq.7.31. The integrand $x^{5/4} Y^3(x)$ in the x-integral of Eq.7.37 has a negative region which, though narrow, actually overwhelms the positive contribution to η_κ from the much longer, but smaller, tail, as we have verified by direct numerical integration[38].

This negative region is purely an artifact of calculating in a *fractional* intrinsic dimension $D = 5/2$. In $D = 2$ for a phantom membrane, where there is no relevant anomalous elasticity for $\vec{h}$, and hence we can calculate $\vec{h}$-$\vec{h}$ correlation functions exactly, we find the analog of Eq.7.31 is

$$G(x, y) = \int \frac{dq_x dq_y}{(2\pi)^2} \frac{e^{iq_x x + iq_y y} q_y^2}{q_x^2 + q_y^4} , \qquad (6.89)$$

$$= \frac{1}{4(\pi|x|)^{1/2}} e^{-y^2/(4|x|)} , \qquad (6.90)$$

which, *unlike* the analogous correlation function in $D = 5/2$, Eq.7.31, is positive definite. Thus, the anomalous contribution to κ in $D = 2$ will also be positive, as we expect on physical grounds (i.e., the shell theory argument summarized in Eq.6.66), while the $5/2 - D$-expansion is *qualitatively* wrong in predicting a negative renormalization of κ. Clearly, it cannot be trusted quantitatively either, and is, in fact, totally useless.

(II)

23

Direct, uncontrolled RG in $D = 2$. Now, we at least get qualitatively correct upward renormalization of κ. However, here we have a different problem, that appears in *any* perturbative calculation away from an upper critical dimension (and is usually "swept under the rug"): even though $D = 2$ would not, a priori, *appear* to be far below $D = 5/2$, it is, in the sense that graphs that *only* diverge logarithmically in $D = 5/2$ diverge *extremely* strongly in $D = 2$. In particular, following very closely the manipulations that lead to Eq.7.35, we find a contribution to κ of the form

$$\delta\kappa = c_1 \int_0^{c_2 q_y^{-1}} y\, dy \int_0^\infty dx \frac{e^{-3/(4x)}}{x^{3/2}} , \qquad (6.91)$$

where c_1 is a well-determined constant that we could calculate, and c_2 is an *arbitrary* constant which depends on precisely how the infrared divergence of the above integral is cutoff with q_y. This arbitrary constant is the problem: *if* the integral Eq.6.91 *had* diverged logarithmically, the precise value of the constant c_2 would be unimportant (it would just lead to a finite additive constant). But, since the integral in Eq.6.91 diverges so strongly (like $(c_2/q_y)^2$) in $D = 2$, it is *extremely* sensitive to the precise value of c_2, which we have *no clue* as how to choose. Thus, we have *no ability* to predict η_κ at all by this approach.

This strong divergence indicates that in this sense $D = 2$ is quite far from $D = 5/2$, and *any* kind of perturbative approach, even to simply calculating one loop constants like C_g and C_κ, is doomed.

C. Gaussian variational theory of self-avoiding tubules

Here we study the effects of self-avoidance within the tubule phase using the Gaussian variational method, which was previously applied to the study of self-avoidance in crumpled isotropic membranes[21,22] and in polymers[37]. It is important to emphasize that both Flory theory and the Gaussian variational method are uncontrolled approximations in that there is no way to systematically estimate and reduce the error.

We begin with the effective Hamiltonian that describes the long wavelength behavior of the tubule for $d < d_*$.

$$H = \frac{1}{2} \int d^{D-1}x_\perp dy \left[\kappa(\partial_y^2 \vec{h})^2 + t(\partial_\alpha^\perp \vec{h})^2 + \frac{1}{4}g_y(\partial_y \vec{h})^4 \right]$$
$$+ v \int dy\, d^{D-1}x_\perp d^{D-1}x'_\perp \, \delta^{(d-1)}\big(\vec{h}(\mathbf{x}_\perp, y) - \vec{h}(\mathbf{x}'_\perp, y)\big) , \qquad (6.92)$$

where, in contrast to other sections we use the notation H to distinguish long-wavelength effective Hamiltonian (the free energy *functional*) from the actual free energy F. Computation of correlation functions in the presence of the self-avoiding nonlinearity cannot be done exactly.

However, we can replace the Hamiltonian H, Eq.6.92 by a variational Hamiltonian H_v, quadratic in the fields $\vec{h}(\mathbf{x}_\perp, y)$, which allows exact calculations of any correlation function. Following the standard variational procedure, we then pick the "best" form of this variational Hamiltonian, where by "best" we mean that it minimizes an upper bound on the true free energy F:[39]

$$F \le \tilde{F} \equiv \langle H - H_v \rangle_v + F_v . \qquad (6.93)$$

We take our variation ansatz Hamiltonian to be

$$H_v = \frac{1}{2} \int \frac{dk_y d^{D-1}\mathbf{k}_\perp}{(2\pi)^D} G_v(\mathbf{k}_\perp, k_y) |\vec{h}(\mathbf{k}_\perp, k_y)|^2 , \quad (6.94)$$

where $G_v(\mathbf{k}_\perp, k_y)$ is the variational kernel to be optimized over. Note that because of anisotropy intrinsic to the tubule, $G_v(\mathbf{k}_\perp, k_y)$ is *not* rotationally invariant as it is for the analogous analysis of the crumpled phase.

We now compute the right-hand-side of Eq.6.93 and minimize it over $G_v(\mathbf{k}_\perp, k_y)$.

$$\langle H - H_v \rangle_v = \frac{A}{2} \int_k \left(\kappa k_y^4 + t k_\perp^2 - G_v(\mathbf{k}_\perp, k_y) \right) \langle |\vec{h}(\mathbf{k}_\perp, k_y)|^2 \rangle_v$$
$$+ \frac{g_y}{8} \int_\mathbf{x} \langle (\partial_y \vec{h})^4 \rangle_v + v \int_\mathbf{x} \left\langle \delta^{(d-1)}\big(\vec{h}(\mathbf{x}_\perp, y) - \vec{h}(\mathbf{x}'_\perp, y)\big) \right\rangle_v , \qquad (6.95)$$

where $A = L_y L_\perp^{D-1}$ is the "area" of the membrane and we defined $\int_\mathbf{x} \equiv \int dy\, d^{D-1}x_\perp d^{D-1}x'_\perp$, and $\int_k \equiv \int dk_y d^{D-1}\mathbf{k}_\perp/(2\pi)^D$.

The above averages are easily evaluated with $\langle |\vec{h}(\mathbf{k}_\perp, k_y)|^2 \rangle_v = (d-1)/G_v(\mathbf{k})$ and $\langle \delta \rangle_v \equiv \langle \delta^{(d-1)}\big(\vec{h}(\mathbf{x}_\perp, y) - \vec{h}(\mathbf{x}'_\perp, y)\big) \rangle_v$ given by

$$\langle \delta \rangle_v = \left\langle \int \frac{d^{d-1}\vec{q}}{(2\pi)^{d-1}} e^{i\vec{q}\cdot\big(\vec{h}(\mathbf{x}_\perp, y) - \vec{h}(\mathbf{x}'_\perp, y)\big)} \right\rangle_v ,$$
$$= \int \frac{d^{d-1}\vec{q}}{(2\pi)^{d-1}} e^{-q^2 K(|\mathbf{x}_\perp - \mathbf{x}'_\perp|)} ,$$
$$= \frac{1}{(2\pi)^{d-1}} \left(\frac{\pi}{K(|\mathbf{x}_\perp - \mathbf{x}'_\perp|)} \right)^{(d-1)/2} , \qquad (6.96)$$

where,

$$K(|\mathbf{x}_\perp|) = \frac{1}{2(d-1)} \langle |\vec{h}(\mathbf{x}_\perp, 0) - \vec{h}(\mathbf{0}_\perp, 0)|^2 \rangle_v ,$$
$$= \int_k \frac{\big(1 - \cos(\mathbf{k}_\perp \cdot \mathbf{x}_\perp)\big)}{G_v(\mathbf{k})} , \qquad (6.97)$$

and we have used in Eq.6.96 the Fourier representation of the $d - 1$-dimensional delta-function.

Putting all this together we obtain for the right-hand-side of Eq.6.93

$$\frac{\tilde{F}}{(d-1)A/2} = \int_k \left(\frac{\kappa k_y^4 + t k_\perp^2}{G_v(\mathbf{k})} - 1 \right) + \frac{g_y(d+1)}{4(d-1)} \left(\int_k \frac{k_y^2}{G_v(\mathbf{k})} \right)^2$$
$$+ \frac{4v}{(2\pi)^{d-1}(d-1)} \int d^{D-1}x_\perp \left(\frac{\pi}{K(x_\perp)} \right)^{(d-1)/2}$$
$$+ \int_k \log\left(G_v(\mathbf{k}) \right) , \tag{6.98}$$

which when minimized with respect to $G_v(\mathbf{k})$, $\delta \tilde{F}/\delta G_v(\mathbf{k}) = 0$ gives an integral equation

$$G_v(\vec{k}) = \kappa k_y^4 + t k_\perp^2 - \frac{2v}{(4\pi)^{(d-1)/2}} \int \frac{d^{D-1}x_\perp \left(1 - \cos(\mathbf{k}_\perp \cdot \mathbf{x}_\perp)\right)}{K(x_\perp)^{(d+1)/2}} . \tag{6.99}$$

The only effect of the anharmonic elasticity term g_y is to generate an upward renormalization of the effective tension along the y-axis

$$\delta t_y = \frac{g_y(d+1)}{2(d-1)} \int_k \frac{k_y^2}{G_v(\mathbf{k})} . \tag{6.100}$$

Since we must choose the *renormalized* tension along the extended tubule axis (y) to be exactly zero in order to treat the free tubule, all of the anharmonic elastic effects disappear in this Gaussian variational approximation. That is, to correctly model a tubule with free boundaries, we should have started with an elastic Hamiltonian with a bare, negative tension piece that exactly cancelled thermally generated positive contribution in Eq.6.100.

The simultaneous integral equations Eq.6.97 and Eq.6.99 determine $G_v(\mathbf{k})$ and $K(x_\perp)$. At long length scales they are solved by $K(x_\perp) \sim x_\perp^{2\nu}$, where from the definition Eq.6.97, we see that $K(x_\perp = L_\perp)$ is proportional to the square of the radius of gyration or the tubule thickness that we are after, and hence the ν that solves these coupled non-linear integral equations will be the Gaussian variational prediction for the radius of gyration exponent as well. We substitute this scaling ansatz into Eq.6.99 for $G_v(\vec{k})$, and find that while for $d > d_{uc}^{SA}$ the self-avoidance is irrelevant and $\nu = (5 - 2D)/4$ (as found in Sec.VI A), for $d < d_{uc}^{SA}$ these integral equations can only be solved if the $t k_\perp^2$ term in Eq.6.99 is exactly cancelled by a part coming from the integral in the last term and the resulting propagator takes the form

$$G_v(\mathbf{k}) = \kappa k_y^4 + \tilde{v} k_\perp^{(d+1)\nu - D + 1} , \tag{6.101}$$

where $\tilde{v} \propto v$ is an effective self-avoiding interaction parameter. Substituting this form into Eq.6.97 for $G_v(\mathbf{k})$, and requiring self-consistency with our original ansatz $K(x_\perp) \sim x_\perp^{2\nu}$ gives

$$x_\perp^{2\nu} \propto \int \frac{d^{D-1}q_\perp \, dq_y \left(1 - \cos(\mathbf{k}_\perp \cdot \mathbf{x}_\perp)\right)}{\kappa k_y^4 + \tilde{v} k_\perp^{(d+1)\nu - D + 1}} . \tag{6.102}$$

Making the change of variables $\mathbf{q}_\perp \equiv \tilde{\mathbf{q}}_\perp / |\mathbf{x}_\perp|$ and $q_y \equiv \tilde{q}_y / |\mathbf{x}_\perp|^\alpha$, with $\alpha = (\nu(d+1) - D + 1)/4$ reveals

that the right hand side of Eq.6.102 is proportional to $x_\perp^\gamma$ with

$$\gamma = 1 - D + 3\alpha ,$$
$$= \frac{7(1 - D) + 3\nu(d+1)}{4} . \tag{6.103}$$

To satisfy the self-consistent condition Eq.6.102, γ must be equal to 2ν. The resulting simple linear equation for ν has a solution (for $d < d_{uc}^{SA}(D)$):

$$\nu = \frac{7D - 7}{3d - 5} , \tag{6.104}$$

which for the physical case of $D = 2$ gives

$$\nu = \frac{7}{3d - 5} , \quad \text{for} \quad d < 11 , \tag{6.105}$$
$$= \frac{1}{4} , \quad \text{for} \quad d \geq 11 \tag{6.106}$$

We observe that $\nu(d = 4) = 1$, and therefore (according to the Gaussian variational approximation) the tubule is no longer crumpled along the $\perp$-direction. This suggests that the tubule phase is unstable to the flat phase in embedding dimensions $d < 4$ (which unfortunately includes the physical case of $d = 3$). However, as discussed in the Introduction, the Gaussian variational method is an uncontrolled approximation. It probably does give the correct *trends* of, e.g., exponents with dimensionality d. However, the variational approach is very close, in spirit and technically, to the large d expansion methods, and therefore intrinsically unable to obtain the small d dependence correctly. It is therefore difficult to place any faith in the actual values of exponents, particularly when the value of ν at small d actually determines whether the tubule phase survives or not.

We believe that this Gaussian variation theory is incorrect in predicting that the tubule phase does not exist in the presence of self-avoidance in $d = 3$, and reiterate our earlier observation that both Flory theory[1] and the $\epsilon = 11 - d$-expansion[3] predict that the tubule phase survives self-avoidance. Since both these latter approaches agree quite closely with each other, and since, furthermore, the ϵ-expansion is the only controlled approximation, we are far more inclined to trust them than the uncontrolled Gaussian approximation, which agrees with neither.

The final determination of whether or not the tubule phase survives self-avoidance will, or course, rest upon simulations and experiments, both of which we hope our analytic work stimulates.

VII. FLUCTUATION EFFECTS AT CRUMPLED-TO-TUBULE AND TUBULE-TO-FLAT TRANSITIONS

The transition from the crumpled-to-flat phase in isotropic membranes has been previously studied[24] and is

25

predicted to be driven first order by fluctuations for embedding dimensions $d < d_c = 219$. As can be seen from Fig.1 this direct transition is very special for anisotropic membranes. It is easy to see that any path finely tuned to pass through the tetracritical point will undergo a direct crumpled-to-flat transition identical to that of isotropic membranes, discussed in Ref. 24.

Here we focus on the new transitions crumpled-to-tubule and tubule-to-flat, which are generic for membranes with *any* amount of anisotropy. As we discussed at the end of Sec.III, there are two possible mean field phase diagram topologies depending on the values of microscopic elastic moduli of the membrane. However, for the crumpled-to-tubule transition there is no difference. In this section we first study the crumpled-to-tubule transition for a phantom membrane using a detailed renormalization group analyses. We then study both the crumpled-to-tubule and tubule-to-flat transition using scaling theory, incorporating the effects of both the anharmonic elasticity and self-avoidance. We postpone the more technically challenging renormalization group analysis of the phantom tubule-to-flat transition[41] and renormalization group analysis of crumpled-to-tubule and tubule-to-flat transitions for self-avoiding membranes[33] for future publications.

A. Renormalization group analysis of crumpled-to-tubule transition

We start out with the general free energy defined in Eq.2.1, for now ignoring the self-avoiding interaction. Without loss of generality we will study the transition from the crumpled to the y-tubule phase. As discussed above, in mean-field theory, this transition occurs when $t_y \to 0$ from above, while $t_\perp$ remains finite and positive. Hence, simple power-counting on the quadratic part of the free energy leads to anisotropic scaling at the transition with $q_\perp \propto q_y^2$. Therefore, the only relevant terms quadratic in $\vec{r}$ near the transition are: the bending rigidity along the y-direction ($\kappa_y \left(\partial_y^2 \vec{r}\right)^2$), and the surface tension terms along the y and $\perp$-directions $t_y \left(\partial_y \vec{r}\right)^2$ and $t_\perp \left(\partial_\alpha^\perp \vec{r}\right)^2$, respectively. The corresponding noninteracting propagator at the transition is

$$\langle r_i(\mathbf{q}) r_j(-\mathbf{q}) \rangle = \frac{\delta_{ij}}{t_\perp q_\perp^2 + t_y q_y^2 + \kappa_y q_y^4} \equiv C(\mathbf{q}) \delta_{ij} , \quad (7.1)$$

The anisotropic scaling dictated by this noninteracting propagator at the transition ($t_y = 0$) leads to significant simplification of the interaction term in the free energy. Keeping only the dominant nonlinearity we obtain

$$F[\vec{r}(\mathbf{x})] = \frac{1}{2} \int d^{D-1}x_\perp dy \left[\kappa_y \left(\partial_y^2 \vec{r}\right)^2 + t_\perp \left(\partial_\alpha^\perp \vec{r}\right)^2 \right.$$
$$\left. + t_y \left(\partial_y \vec{r}\right)^2 + \frac{u_{yy}}{2} \left(\partial_y \vec{r} \cdot \partial_y \vec{r}\right)^2 \right] . \quad (7.2)$$

The critical properties of the crumpled-to-tubule transition can be obtained by applying scaling theory and the renormalization group to this free energy exactly as we did earlier in treating fluctuations in the tubule phase itself. In this case, "lengths" means intrinsic coordinates $\mathbf{x} = (\mathbf{x}_\perp, y)$, and the "fields" are the extrinsic positions $\vec{r}(\mathbf{x})$. Because of the strong *scaling* anisotropy of the quadratic pieces of the free energy, we rescale $\mathbf{x}_\perp$ and y anisotropically:

$$\mathbf{x}_\perp = \mathbf{x}'_\perp e^l , \quad (7.3)$$
$$y = y' e^{zl} , \quad (7.4)$$

and rescale the "fields" according to

$$\vec{r}(\mathbf{x}) = e^{\chi l} \vec{r}'(\mathbf{x}') . \quad (7.5)$$

Under this transformation

$$\kappa_y(l) = \kappa_y e^{(D-1-3z+2\chi)l} , \quad (7.6)$$
$$t_\perp(l) = t_\perp e^{(D-3+z+2\chi)l} , \quad (7.7)$$

Requiring that both κ_y and $t_\perp$ remain fixed under this rescaling (zeroth order RG transformation) fixes the "anisotropy" exponent z and the "roughness" exponent χ (which is the analog of ν for the tubule phase):

$$z = \frac{1}{2} , \quad (7.8)$$
$$\chi = (5/2 - D)/2 . \quad (7.9)$$

Although this choice keeps the quadratic (in $\vec{r}$) part of F Eq.7.2) unchanged, it *does* change the quartic piece:

$$u_{yy}(l) = u_{yy} e^{(D-1-3z+4\chi)l} , \quad (7.10)$$
$$= u_{yy} e^{(5/2-D)l} , \quad (7.11)$$

where in the second equality we have used the results Eqs.7.8 and 7.9 for z and χ. We see that for $D < 5/2$, u_{yy} grows upon rescaling. Physically, this means that its effects become more important at longer length scales. At sufficiently long length scales, it completely invalidates the harmonic elastic theory and the naive perturbation theory in the nonlinearity u_{yy} around it, even for arbitrarily small coupling u_{yy}. Simple additional anisotropic rescaling of $\mathbf{x}_\perp = \alpha \mathbf{x}'_\perp$ and $y = \beta y'$, with $\beta = (t_\perp/\kappa_y)^{1/2}\alpha^2$, which rescales κ_y and $t_\perp$ to 1, reveals that the effective coupling constant of the nonlinearity is u_{yy}/κ_y. This, together with Eq.7.11, predicts that the characteristic length scale $L_\perp^{nl}$ beyond which the *dimensionless* coupling constant becomes of order 1 and the harmonic elastic theory and perturbation theory (around it) break down is

$$L_\perp^{nl} = \left(\frac{\kappa_y}{u_{yy}}\right)^{1/(5/2-D)} . \quad (7.12)$$

To analyze the new behavior that prevails on even *longer* length scales requires a full-blown renormalization group analysis.

Such an analysis[36] will lead to corrections to the simple rescaling of κ_y, $t_\perp$, and t_y, due to the non-linearities (in this case u_{yy}, as discussed above). These corrections can be absorbed into "anomalous" exponents η_κ, η_t, and $\delta\theta$, defined by the large renormalization group "time" ($l \to \infty$) limits of $\kappa_y(l)$, $t_\perp(l)$, and $t_y(l)$, respectively:

$$\kappa_y(l) = \kappa_y e^{(D-1-3z+z\eta_\kappa+2\chi)l} \ , \tag{7.13}$$

$$t_\perp(l) = t_\perp e^{(D-3+z+\eta_t+2\chi)l} \ , \tag{7.14}$$

$$t_y(l) = t_y e^{(D-1-z-\delta\theta+2\chi)l} \equiv t_y e^{\lambda_t l} \ , \tag{7.15}$$

The exponent λ_t defined above is the thermal eigenvalue of the reduced temperature (surface tension along y-direction) which is an inverse of the correlation length exponent along the $\perp$-direction (see below). Requiring that κ_y and $t_\perp$ remain invariant under the renormalization group transformation determines the values the anisotropy exponent z and the field rescaling exponent χ,

$$z = \frac{2 - \eta_t}{4 - \eta_\kappa} \ , \tag{7.16}$$

$$\chi = \frac{10 - 4D + \eta_\kappa(D - 3 + \eta_t) - 3\eta_t}{8 - 2\eta_\kappa} \ ,$$

which, as quoted above in Eqs.7.8 and 7.9, reduce to $z = 1/2$ and $\chi = (5/2 - D)/2$, for $\eta_\kappa = \eta_t = 0$, as is valid at zero order in perturbation theory in u_{yy}.

Once the values of η_t, η_κ and χ at the critical point are determined, the renormalization group gives a relation between correlation functions at or near criticality (small t_y) and at small wavectors (functions that are difficult to compute, because direct perturbation theory is divergent) to the same correlation functions away from criticality and at large wavevectors (functions that can be accurately computed using perturbation theory). For example the behavior of the correlation lengths near the transition can be deduced in this way:

$$\xi_\perp(t_y) = e^l \xi_\perp(t_y e^{\lambda_t l}) \ , \tag{7.17}$$

$$\xi_y(t_y) = e^{zl} \xi_y(t_y e^{\lambda_t l}) \ , \tag{7.18}$$

where in the above we assumed that a critical fixed point exists and all other coupling constants have well-defined values at the fixed point. Using the above equations for $t_y e^{\lambda_t l} \approx 1$, we obtain,

$$\xi_\perp(t_y) \approx a \, t_y^{-\nu_\perp} \ , \tag{7.19}$$

$$\xi_y(t_y) \approx a \, t_y^{-\nu_y} \ ,$$

where $a \approx \xi(1)$ is the microscopic cutoff and,

$$\nu_\perp = \frac{1}{\lambda_t} = \frac{4 - \eta_\kappa}{2(2 - \eta_t - 2\delta\theta) - \eta_\kappa(2 - \eta_t - \delta\theta)} \ , \tag{7.20}$$

$$\nu_y = z\nu_\perp \ . \tag{7.21}$$

We now compute the anomalous exponents to lowest non-zero order in ϵ, where $\epsilon = 5/2 - D$. As usual in the ϵ-expansion, the order at which a given graphical correction enters the perturbation theory is equal to the number of loops in the associated Feynman graph. We split the field $\vec{r}(\mathbf{x})$ into short and long wavevector parts $\vec{r}(\mathbf{x}) = \vec{r}_<(\mathbf{x}) + \vec{r}_>(\mathbf{x})$ and integrate over the fast fields $\vec{r}_>(\mathbf{x})$. Diagrammatically this leads to one-loop corrections to u_{yy} and t_y. There are no corrections to κ_y to first order in ϵ, i.e., $\eta_\kappa = O(\epsilon^2)$. Furthermore, since the interaction u_{yy} always carries a factor of q_y with every field $\vec{r}$, the $t_\perp$ tension remains unrenormalized, and $\eta_t = 0$ to all orders, implying $z = 1/2 + O(\epsilon^2)$ and $\chi = (5/2 - D)/2 + O(\epsilon^2)$.
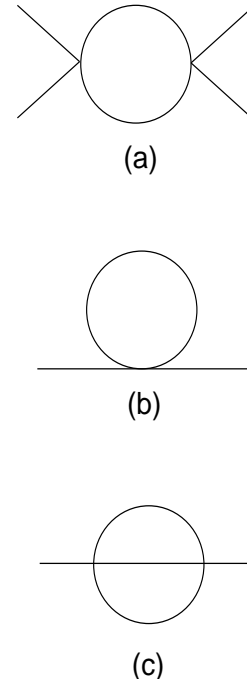
(a)

(b)

(c)

FIG. 11. Feynman graphs that renormalize: (a) the non-linearity u_{yy}, (b) the tension t_y and (c) the bending rigidity κ_y.

The first two diagrams in Fig.11, followed by the rescaling introduced above (necessary to restore the original uv cutoff), lead to the one-loop recursion relations for $u = (K_{3/2}/\sqrt{2})u_{yy}/(\kappa_y^{5/4}t_\perp^{3/4})$ and t_y, respectively,

$$\frac{\partial u}{\partial l} = \epsilon u - (d + 8)u^2 \ , \tag{7.22}$$

$$\frac{\partial t_y}{\partial l} = (1 - (d + 2)u)t_y \ , \tag{7.23}$$

27

where $\epsilon = 5/2 - D$ and $K_{3/2}$ is the surface of area of a 3/2-dimensional sphere divided by $(2\pi)^{3/2}$. As usual, in the above, we also redefined t_y to be the reduced temperature, measured from its true value at the transition (which in mean-field theory starts out at 0, but is shifted to a negative value by fluctuations). Note that, in contrast to the familiar $\epsilon = 4 - D$- expansion for critical phenomena, for which $\epsilon = 1$ in the physical case $D = 3$, here we have $\epsilon = 1/2$ in the physical case $D = 2$. Hence, our one-loop expansion should be quantitatively more accurate by a factor of $\epsilon^{-2} = 4$, than the $4 - D$-expansion at the same order. Thus, we expect our one-loop values for $\nu_\perp$ and ν_y to be accurate to ± 0.02.

Examining Eq.7.23, we observe that for $D < D_{uc} = 5/2$, (i.e., for $\epsilon > 0$) the Gaussian fixed point is unstable and the critical properties of the crumpled-to-tubule transition are characterized by a nontrivial fixed point with a fixed point value u_* of u given by:

$$u_* = \frac{\epsilon}{d+8} . \qquad (7.24)$$

Note that, in contrast to the treatment of crumpled-to-flat transition in isotropic membranes[24], where the critical point was only stable for an unphysically large value of the embedding dimension $d > 219$, the critical point characterizing the crumpled-to-tubule transition found here is stable for all d.

Equation 7.23 can be easily integrated once the fixed point value u_*, Eq.7.24, is inserted for u; comparison with the general Eq.7.16 then gives λ_t:

$$\lambda_t = 1 - \left(\frac{d+2}{d+8}\right)\epsilon , \qquad (7.25)$$

which, upon using Eqs.7.20 and 7.21, gives for a physical membrane ($D = 2$, $d = 3$)

$$\nu_\perp \approx 1.227 , \qquad (7.26)$$
$$\nu_y \approx 0.614 . \qquad (7.27)$$

The η_κ exponent to $O(\epsilon^2)$ is determined by the diagram in Fig.11c. Evaluating this diagram in real space and then Fourier transforming, we find that this contributes to the free energy

$$\delta F = -16u^2(d+2)\int_q q_y^2 |\vec{r}(\mathbf{q})|^2 \Gamma(\mathbf{q}) , \qquad (7.28)$$

where

$$\Gamma(\mathbf{q}) \equiv \int d^{3/2}\mathbf{x}_\perp dy \, e^{i\mathbf{q}\cdot\mathbf{x}} \, G^3(\mathbf{x}) , \qquad (7.29)$$

with, in turn,

$$G(\mathbf{x}_\perp, y) = \int \frac{d^{3/2}\mathbf{q}_\perp dq_y}{(2\pi)^{5/2}} \frac{e^{i\mathbf{q}_\perp\cdot\mathbf{x}_\perp} e^{iq_y y} q_y^2}{q_\perp^2 + q_y^4} , \qquad (7.30)$$

where we have rescaled lengths so that $\kappa_y = t_\perp = 1$.

After a contour integral over q_y, and an angular integral $\int_0^\pi d\theta (\sin\theta)^{(D-2)} e^{i\mathbf{q}_\perp \cdot \mathbf{x}_\perp}$, we obtain,

$$G(\mathbf{x}_\perp, y) = 2^{-7/4}\pi^{-3/4}y^{-2}\left(\frac{x_\perp}{y^2}\right)^{1/4} Y\left(\frac{x_\perp}{y^2}\right) , \qquad (7.31)$$

where we have defined

$$Y(x) \equiv \int_0^\infty du \, u^{1/4} J_{-1/4}(xu) \, e^{-\sqrt{u/2}} \cos(\sqrt{u/2} + \pi/4) . \qquad (7.32)$$

Now going back to Eq.7.28 we observe that the $\mathbf{q} = \mathbf{0}$ piece of $\Gamma(\mathbf{q})$ contributes to the $q_y^2 |\vec{r}(\mathbf{q})|^2$ part of F, additively renormalizing t_y which corresponds to the usual inconsequential T_c (critical tension) shift. The order q_y^2 piece of $\Gamma(\mathbf{q})$ renormalizes κ_y. We define

$$\Gamma(\mathbf{q}) = \Gamma(0) - \frac{1}{2}q_y^2 B(q_y) , \qquad (7.33)$$

where

$$B(q_y) \equiv \int_{\Lambda^{-1} < |y| < q_y^{-1}} dy \, d^{3/2}x_\perp \, y^2 G^3(\mathbf{x}_\perp, y) . \qquad (7.34)$$

Note that the infra-red cutoff on the integral over y is q_y^{-1}. This integral diverges logarithmically as $q_y \to 0$. We can identify the coefficient of the logarithm with η_κ in the expression $\kappa_y(q_y) \propto q_y^{-\eta_\kappa}$.

To extract this logarithmic divergence, we make a change of variables in the integral $|\mathbf{x}_\perp| \equiv xy^2$ and find

$$B(q_y) \equiv -\frac{1}{2^{13/4}\pi^{3/2}\Gamma(3/4)}\int_{\Lambda^{-1}}^{q_y^{-1}} \frac{dy}{y} \int_0^\infty dx \, x^{5/4} \, Y^3(x) , \qquad (7.35)$$

where we have used the fact that the surface area of a 3/2-dimensional sphere is $2\pi^{3/2}/\Gamma(3/4)$, and taken into account the factor of 2 coming from the fact that the original integral over y extends over both $y > 0$ and $y < 0$.

Putting all of the above together and evaluating the coefficient of the $\log(q_y)$ at the fixed point value of g_y^* from Eq.7.24, we obtain

$$\eta_\kappa = \frac{C(2)(d+2)}{8(d+8)^2}\epsilon^2 , \qquad (7.36)$$

where

$$C(2) \equiv 2^{23/4}\Gamma(3/4)\int_0^\infty dx \, x^{5/4} \, Y^3(x) , \qquad (7.37)$$

The value of $C(2)$ has been calculated numerically[40] to be $C(2) \approx -1.166 \pm 0.001$. Using this value, $\epsilon = 1/2$, and $d = 3$ in Eq.7.36, we find that η_κ is very small,

$$\eta_\kappa(D = 2, d = 3) \approx -0.0015 . \qquad (7.38)$$

As noted earlier in our discussion of the tubule phase itself, we do not trust this negative value of η_κ, but, rather, believe it to be an artifact of the peculiar negative regime that appears in the correlation function $G(\mathbf{x}_\perp, y)$ in $D = 5/2$. We expect η_κ to be positive, but still quite small, at the *phantom* tubule-to-crumpled transition.

Given the smallness of η_κ and ϵ, and the vanishing of η_t, the exponents computed here to first order in ϵ are expected to be very accurate.

B. Scaling theory of crumpled-to-tubule and tubule-to-flat transitions

We will now incorporate the effects of self-avoidance on these transitions. We have not yet done the full renormalization group analysis of this problem (which must include *both* the elastic and self-avoiding interaction nonlinearities)[33], and limit ourselves here to discussing scaling theory and the Flory approximation.

Near the crumpled-to-tubule transition, for square membranes of internal size L, we make the following general scaling ansatz for the extensions R_y and R_G of the membrane along and orthogonal to the tubule axis, respectively:

$$R_{G,y} = L^{\nu_{ct}^{G,y}} f_{G,y}(t_y L^\phi) ,$$

$$\propto \begin{cases} t_y^{\gamma_+^{G,y}} L^{\nu_c}, & t_y > 0, L >> \xi_{ct} \\ L^{\nu_{ct}^{G,y}}, & L << \xi_{ct} \\ |t_y|^{\gamma_-^{G,y}} L^{\nu_t^{G,y}}, & t_y < 0, L >> \xi_{ct} \end{cases} \quad (7.39)$$

where subscripts t, c and ct refer to tubule, crumpled and tubule-to-crumpled transition, respectively, and $\xi_{ct} \propto |t_y|^{-1/\phi}$ is a correlation length for the crumpled-to-tubule transition, $t_y = (T - T_{ct})/T_{ct}$, T_{ct} is the crumpled-to-tubule transition temperature, and $t_y > 0$ corresponds to the crumpled phase.

Note that we have built into the scaling laws the fact that both R_y and R_G scale like L^{ν_c} in the crumpled phase, with ν_c the radius of gyration exponent for the crumpled phase (which, as noted earlier, is the same for anisotropic and isotropic membranes). Due to the extended nature of the tubule phase, $\nu_t^y = 1$, of course. The anisotropy in manifested in the crumpled phase only through the different temperature dependences of R_G and R_y. The former of these vanishes as $t_y \to 0^+$ (since the radius of gyration in the tubule phase is much less than that in the crumpled phase, since $\nu_t < \nu_c$), which implies $\gamma_+^G > 0$, while the latter diverges as $t_y \to 0^+$, since the tubule ultimately extends in that direction, which implies $\gamma_+^y < 0$.

Note also that our expression Eq.7.39, and, in particular, the fact that $R_G \neq R_y$ *even* above the crumpled-to-tubule transition (i.e., *in* the *crumpled* phase), implies a spontaneous breaking of rotational invariance even in the crumpled phase! This seemingly bizarre (but correct) result is actually not all that unfamiliar: polymers,

which are always crumpled, nonetheless assume, on average, non-spherical shapes[42], as can be seen, for example, by looking at the ratio of the average maximum and minimum eigenvalues of the moment of inertia tensor. Our result Eq.7.39 for $t_y > 0$ is only a little more surprising, since it predicts an aspect ratio R_y/R_G that actually *diverges* as $T \to T_{ct}^+$, and membrane begins to extend into a tubule configuration.

The exponents $\gamma_{+/-}^{G,y}$ defined in above equation obey the scaling laws

$$\gamma_+^{G,y} = \frac{\nu_c - \nu_{ct}^{G,y}}{\phi} , \quad (7.40)$$

$$\gamma_-^{G,y} = \frac{\nu_t^{G,y} - \nu_{ct}^{G,y}}{\phi} . \quad (7.41)$$

As always, these scaling laws follow from requiring that the generalized scaling form matches on to known results in the appropriate limits.

From Flory theory, we can derive the values of the critical exponents in Eq.7.39, as we have already derived the exponents ν_t and ν_c characterizing the tubule and crumpled phases, simply by being more careful about temperature dependent factors in that derivation. Again, we start by estimating the total self-avoidance energy Eq.2.1 in the tubule phase (i.e., $t_y < 0$) as $E_{SA} \approx V\rho^2$. Now, however, we very carefully write the volume V in the embedding space occupied by the tubule as $V \approx R_G^{d-1}R_y$. Writing

$$R_y = \zeta_y L_y , \quad (7.42)$$

as we did earlier in our discussion of mean field theory in the *absence* of self-avoidance, and using $\rho = M/V$ for the embedding space density of the tubule, and again using the fact that the tubule mass $M \approx L_\perp^{D-1}L_y$, we see that

$$E_{SA} \approx v\frac{L_y L_\perp^{2(D-1)}}{\zeta_y R_G^{d-1}} . \quad (7.43)$$

Using this estimate of the self-avoidance energy in Eq.2.1, and estimating the other terms in that expression by scaling, we obtain the full Flory theory for the tubule phase, with all temperature dependent effects taken (admittedly crudely) into account:

$$E_{FL} = \left(t_y\zeta_y^2 + u_{yy}\zeta_y^4 + t_\perp\left(\frac{R_G}{L_\perp}\right)^2 \right) L_\perp^{D-1}L_y + v\frac{L_y L_\perp^{2(D-1)}}{\zeta_y R_G^{d-1}} . \quad (7.44)$$

Minimizing this over R_G, we obtain

$$R_G \approx L_\perp^{\nu_t}\left(\frac{v}{t_\perp \zeta_y}\right)^{1/(d+1)} , \quad (7.45)$$

where, as we found earlier, $\nu_t = \frac{D+1}{d+1}$, but now we have the singular temperature dependence of R_G near the

crumpled-to-tubule transition explicit through the presence of the ζ_y term. Inserting this expression for R_G into Eq.7.44, we find

$$E_{FL} = \left(t_y \zeta_y^2 + u_{yy} \zeta_y^4 + t_\perp^{\frac{d-1}{d+1}} \left(\frac{v}{\zeta_y} \right)^{\frac{2}{d+1}} L_\perp^{-\frac{2(d-D)}{d+1}} \right) L_\perp^{D-1} L_y \ .$$

(7.46)

The exponents defined by Eq.7.39 can now be obtained by minimizing E_{FL} in Eq.7.46 with respect to ζ_y, which amounts to balancing two of the three terms in E_{FL}, which two, depending on whether one is interested in the crumpled phase ($t_y > 0$), the tubule phase ($t_y < 0$), or the transition between them ($t_y = 0$).

In the crumpled phase $t_y > 0$, as a result, the order parameter ζ_y vanishes in the thermodynamic limit, allowing us to neglect the quartic ζ_y^4 term relative to the quadratic ζ_y^2 one. Balancing the remaining two terms

$$t_y \zeta_y^2 \approx t_\perp^{\frac{d-1}{d+1}} \left(\frac{v}{\zeta_y} \right)^{\frac{2}{d+1}} L_\perp^{-\frac{2(d-D)}{d+1}} \ ,$$

(7.47)

we obtain

$$\zeta_y \approx \left(\frac{v^2 t_\perp^{d-1}}{t_y^{d+1}} \right)^{\frac{1}{2(d+2)}} L_\perp^{-\frac{d-D}{d+2}} \ ,$$

(7.48)

Using this expression for ζ_y inside Eq.7.42 for R_y gives, for a square membrane ($L_y = L_\perp = L$)

$$R_y \approx \left(v^2 t_\perp^{d-1} \right)^{\frac{1}{2(d+2)}} t_y^{-\frac{d+1}{2(d+2)}} L_\perp^{\frac{D+2}{d+2}} \ ,$$

(7.49)

which, after comparing with the general form for R_y, Eq.7.39, gives

$$\nu_c = \frac{D+2}{d+2} \ ,$$

(7.50)

$$\gamma_+^y = -\frac{d+1}{2(d+2)} \ ,$$

(7.51)

equation 7.50 being a well-known Flory result for the radius of gyration exponent ν_c for a D-dimensional manifold, embedded in d dimensions,[30–32] and γ_+^y new and special to anisotropic membranes. Furthermore, inserting ζ_y, Eq.7.48 inside Eq.7.45 for R_G, we obtain

$$R_G \approx \left(v^{\frac{d+3}{(d+2)(d+1)}} t_\perp^{-\frac{d+5}{2(d+2)(d+1)}} \right) t_y^{\frac{1}{2(d+2)}} L_\perp^{\frac{D+2}{d+2}} \ , \quad (7.52)$$

which, not surprisingly gives the same expression for ν_c as in Eq.7.50, and predicts

$$\gamma_+^G = \frac{1}{2(d+2)} \ .$$

(7.53)

$\gamma_+^y \neq \gamma_+^G$ supports our earlier claim that even the crumpled phase spontaneously breaks rotational invariance in the embedding space. It does so gently by having the identical growth (for square membranes) of R_G and R_y with L, but exhibiting anisotropy via the prefactors, with the ratio R_y/R_G diverging as the crumpled-to-tubule transition is approached.

The tubule phase is characterized by $t_y < 0$ and a finite order parameter $\zeta_y > 0$. Therefore in this phase, the term proportional to $t_\perp^{d-1/(d+1)}$ in E_{FL}, Eq.7.46, clearly becomes negligible relative to the first two terms when $L_\perp \to \infty$. Therefore, we can neglect that term for a sufficiently large membrane (i.e., a membrane larger than the critical correlation length ξ_{cr}). Minimizing the remaining first two terms in E_{FL} therefore gives $\zeta_y \propto \sqrt{|t_y|}$, (independent of $L_\perp$) as in mean-field theory in the absence of self-avoidance. Inserting this inside R_y, Eq.7.42 and comparing with the general scaling form for R_y, implies for a square membrane

$$\nu_t^y = 1 \ ,$$

(7.54)

$$\gamma_-^y = \frac{1}{2} \ .$$

(7.55)

Using this in the earlier expression Eq.7.45 for R_G, we obtain the last line of Eq.7.39, with

$$\nu_t^G = \frac{D+1}{d+1} \ ,$$

(7.56)

$$\gamma_-^G = -\frac{1}{2(d+1)} \ .$$

(7.57)

(7.58)

Finally, right at the crumpled-to-tubule transition, $t_y = 0$ and we must balance the last two terms in E_{FL}, Eq.7.46. Minimizing E_{FL} over ζ_y, we find at the transition

$$\zeta_y \propto L_\perp^{-\frac{(d-D)}{3+2d}}$$

(7.59)

which, when inserted in Eq.7.42 for R_y implies for a square membrane that

$$R_y \propto L^{\frac{D+d+3}{3+2d}}$$

(7.60)

right at the transition. This leads to

$$\nu_{ct}^y = \frac{D+d+3}{2d+3}$$

(7.61)

for a square membrane. Using the result Eq.7.59 for ζ_y in Eq.7.45 for R_G gives, right at the transition,

$$R_G \propto L_\perp^{\nu_{ct}^G} \ ,$$

(7.62)

with

$$\nu_{ct}^G = \nu_t + \frac{1}{d+1} \left(\frac{d-D}{3+2d} \right) \ ,$$

(7.63)

$$= \frac{2D+3}{2d+3} \ ,$$

(7.64)

30

The scaling relations Eqs.7.40 and 7.41, quoted above, then give

$$\phi = \frac{2(d-D)}{2d+3} \, , \qquad (7.65)$$

and are reassuringly consistent with our independent calculations of exponents $\gamma_{+,-}^{G,y}$, ν_c, $\nu_t^{G,y}$, and $\nu_{ct}^{G,y}$, given in Eqs.7.51,7.53,7.55,7.57, 7.50,7.54,7.56,7.61, and 7.64, above. For the physical case of a two dimensional membrane embedded in a three dimensional space, ($D = 2, d = 3$)

$$\nu_c = 4/5 \, , \qquad (7.66a)$$
$$\nu_{ct}^G = 7/9 \, , \qquad (7.66b)$$
$$\nu_{ct}^y = 8/9 \, , \qquad (7.66c)$$
$$\nu_t = 3/4 \, , \qquad (7.66d)$$
$$\gamma_+^G = 1/10 \, , \qquad (7.66e)$$
$$\gamma_+^y = -2/5 \, , \qquad (7.66f)$$
$$\gamma_-^G = -1/8 \, , \qquad (7.66g)$$
$$\gamma_-^y = 1/2 \, . \qquad (7.66h)$$
$$\phi = 2/9 \, , \qquad (7.66i)$$

Note that the signs of the $\gamma_{+/-}^{G,y}$ imply that R_G shrinks as the crumpled-to-tubule transition is approached from above, and grows as it is approached from below, while R_y does the opposite. Note also that the crumpled-to-tubule transition is quite rounded by finite size effects, even for large membranes, because of the small value of the crossover exponent ϕ, which leads to a large correlation length $\xi_{ct}(t_y)$. Taking an example of a $L = 10\mu m$ membrane with lattice constant $a = 10\text{Å}$, we find that the crumpled-to-tubule transition is rounded at a reduced temperature $t_y \approx (L/a)^{-\phi} \approx 0.13$, while our hypothetical simulation of a 10^4 particle net experiences rounding at $t_y \approx 0.36$. Thus, the transition may not appear sharp experimentally or in simulations, even though it is, in principle, in the thermodynamic limit.

The singular parts of other thermodynamic variables obey scaling laws similar to that for $R_{G,y}$, Eq.7.39. For example the singular part of the specific heat per particle C_v, i.e., a second derivative of the intensive free energy with respect to temperature, is given by

$$C_v \sim \frac{1}{L^D} \frac{\partial^2}{\partial t_y^2} \left(\frac{1}{2} t_y R_y^2 L^{D-2} \right) \, , \qquad (7.67)$$

which, using Eq.7.39 leads to the scaling form for C_v

$$C_v = L^\beta g(t_y L^\phi) \, ,$$
$$\propto \begin{cases} t_y^{-\alpha_+} L^{\beta-\alpha_++\phi}, & t_y > 0, L >> \xi_{ct} \\ L^\beta, & L << \xi_{ct} \\ |t_y|^{-\alpha_-} L^{\beta-\alpha_-\phi}, & t_y < 0, L >> \xi_{ct} \end{cases} \qquad (7.68)$$

where,

$$g(x) \approx \frac{d^2}{dx^2} \left[f_y^2(x) \right] \, . \qquad (7.69)$$

Using the exponents characterizing R_y derived above, we obtain:

$$\beta = 2\nu_{ct}^y - 2 + \phi \, , \qquad (7.70a)$$
$$= 0 \, , \quad \text{Flory theory} \qquad (7.70b)$$

$$\alpha_+ = -2\gamma_+^y + 1 \, , \qquad (7.71a)$$
$$= \frac{2d+3}{d+2} \, , \quad \text{Flory theory} \qquad (7.71b)$$
$$= \frac{9}{5} \, , \quad \text{Flory theory}, \; d = 3 \qquad (7.71c)$$

$$\alpha_- = -2\gamma_-^y + 1 \, , \qquad (7.72a)$$
$$= 0 \, , \quad \text{Flory theory} \qquad (7.72b)$$

This leads to the unusual feature that outside the critical regime (i.e. for $L >> \xi_{ct}$), the singular part of the specific heat above the crumpled-to-tubule transition vanishes in the thermodynamic limit like $L^{-\alpha_++\phi} \sim L^{-2(d-D)/(d+2)} \sim L^{-2/5}$; in the last expression we have used the Flory estimates of the exponents, evaluated in $D = 2$ and $d = 3$. Only within the critical regime does the singular part of the specific heat per particle becomes nonvanishing as $L \to \infty$. Similar results were first found for the direct crumpled-to-flat transition by Paczuski et al.[24].

We now turn to the tubule-to-flat (tf) transition. On both sides of *this* transition, $R_y = L_y \times O(1)$. Therefore only the other two radii of gyration R_x and R_z exhibit critical behavior, which can be summarized by the scaling law:

$$R_{x,z} = L^{\nu_{tf}^{x,z}} f_{x,z}(t_\perp L^{\phi_{tf}}) \, ,$$
$$\propto \begin{cases} t_\perp^{\gamma_+^{x,z}} L^{\nu_t}, & t_\perp > 0, L >> \xi_{tf} \\ L^{\nu_{tf}^{x,z}}, & L << \xi_{tf} \\ |t_\perp|^{\gamma_-^{x,z}} L^{\nu_f^{x,z}}, & t_\perp < 0, L >> \xi_{tf} \end{cases} \qquad (7.73)$$

where $t_\perp = (T - T_{tf})/T_{tf}$, $t_\perp > 0$ is assumed to correspond to the tubule phase, $\xi_{tf} \propto |t_\perp|^{-1/\phi_{tf}}$ is the correlation length for this transition, and the exponents obey the scaling relations

$$\nu_f^z = \zeta \approx 0.59 \, , \qquad (7.74a)$$
$$\nu_f^x = 1 \, , \qquad (7.74b)$$
$$\gamma_+^{x,z} = \frac{\nu_t - \nu_{tf}^{x,z}}{\phi_{tf}} \, , \qquad (7.74c)$$
$$\gamma_-^{x,z} = \frac{\nu_f^{x,z} - \nu_{tf}^{x,z}}{\phi_{tf}} \, . \qquad (7.74d)$$

In the above we have taken the x-direction to be the new (in addition to y) extended direction in the flat phase

(which is why $\nu_f^x = 1$), and ζ is the roughness exponent[28] of the flat phase (quoted for the physical case $D = 2$ and $d = 3$), giving the transverse height fluctuations of the $d - 2$ components of the displacement perpendicular to the flat membrane.

To calculate these exponents, we can use Flory theory in the tubule phase, and at the transition, while in the flat phase, where as discussed above, self-avoidance is irrelevant, we simply match onto the scaling theory[28] of the flat phase. Doing so, we find that Flory theory predicts *identical* behavior for R_x and R_z in the tubule phase and at the transition:

$$\nu_{tf}^x = \nu_{tf}^z = \frac{D+3}{d+3} , \qquad (7.75a)$$

$$= \frac{5}{6} , \quad \text{for} \quad D = 2 , d = 3 , \qquad (7.75b)$$

$$\gamma_+^x = \gamma_+^z = -\frac{1}{d+1} , \qquad (7.75c)$$

$$= -\frac{1}{4} , \quad \text{for} \quad D = 2 , d = 3 . \qquad (7.75d)$$

We believe that the identical temperature ($t_\perp$) and scaling (with L) behavior of R_x and R_z as the tubule-to-flat transition is approached from the tubule side (Eqs.7.75a and 7.75c) is an artifact of Flory theory and that in fact $R_x \gg R_z$ throughout this region, with the ratio R_x/R_z actually diverging as the transition is approached from above. That is, we expect that in reality $\nu_{tf}^x > \nu_{tf}^z$ and $\gamma_+^x < \gamma_+^z$.

In addition, Flory theory predicts

$$\phi_{tf} = \frac{2(d-D)}{d+3} , \qquad (7.76)$$

$$= \frac{1}{3} , \quad \text{for} \quad D = 2 , d = 3 . \qquad (7.77)$$

In the flat phase, γ_-^x follows from simply minimizing the mean field free energy *without* self-avoidance (since self-avoidance is irrelevant in the flat phase), giving

$$\gamma_-^x = \frac{1}{2} , \qquad (7.78)$$

while matching $R_z = L^\zeta |t_\perp|^{\gamma_-^z}$ onto the critical prediction $R_z \propto L^{\nu_{tf}^z}$ at the correlation length $L = \xi = |t_\perp|^{-1/\phi_{tf}}$ gives

$$\gamma_-^z = \frac{\zeta - \nu_{tf}^z}{\phi_{tf}} , \qquad (7.79)$$

$$\approx -0.73 , \qquad (7.80)$$

where the first equality is an exact scaling law, while the second, approximate one uses Flory theory for ϕ_{tf} and ν_{tf}^x, and the SCSA calculation[28] of ζ for the flat phase, all evaluated in the physical case $D = 2$ and $d = 3$.

As the tubule-to-flat transition is approached from below (the flat phase side) R_x shrinks as $R_x \approx |t_\perp|^{1/2}L$ and

R_z increases as $R_z \approx |t_\perp|^{-0.73}L^{0.59}$ with vanishing $|t_\perp|$. Approaching this transition from above (the tubule phase side) R_x and R_z both extend as $R_{x,z} \approx |t_\perp|^{-1/4}L^{3/4}$ with vanishing $t_\perp$ to the $L^{5/6}$ scaling at the tubule-to-flat critical point.

The singular part of the specific heat again obeys a scaling law:

$$C_v = L^{\beta_{tf}} g_{tf}(t_\perp L^{\phi_{tf}}) ,$$

$$\propto \begin{cases} t_\perp^{-\alpha_+^{tf}} L^{\beta_{tf} - \alpha_+^{tf}\phi_{tf}}, & t_\perp > 0, L >> \xi_{tf} \\ L^{\beta_{tf}}, & L << \xi_{tf} \\ |t_\perp|^{-\alpha_-^{tf}} L^{\beta_{tf} - \alpha_-^{tf}\phi_{tf}}, & t_y < 0, L >> \xi_{tf} \end{cases} \quad (7.81)$$

where, in Flory theory,

$$\alpha_+^{tf} = \frac{3}{2} , \qquad (7.82a)$$

$$\alpha_-^{tf} = 0 , \qquad (7.82b)$$

$$\beta_{tf} = 2\nu_{tf} + \phi_{tf} - 2 = 0 , \qquad (7.82c)$$

Thus, again, the singular part of the specific heat vanishes (now like $L^{-1/2}$) in the thermodynamic limit above (i.e., on the tubule side) of the transition, while it is $O(1)$ and smooth as a function of temperature in both the critical regime and in the flat phase.

VIII. SUMMARY AND CONCLUSIONS

In summary, we have studied the effects of intrinsic anisotropy in polymerized membrane. We found that this seemingly innocuous generalization leads to a wealth of new phenomena, most remarkable of which is that *any* amount of anisotropy leads to a new, tubule phase which intervenes between the previously predicted flat and crumpled phases in anisotropic membranes (See Fig.1). We have presented a detailed theory of the anisotropic membrane focusing on the new tubule phase. Considering thermal fluctuations in the tubule phase we have shown that the *phantom* tubule phase exhibits anomalous elasticity, and calculated the elasticity and size exponents *exactly*, as summarized in Eqs.5.15,5.16,5.38,5.48. We then considered the physically more relevant case of a *self-avoiding* tubule, finding that self-avoiding interaction is important for physical dimensionalities. Establishing relations between the exponent characterizing the diameter of the tubule and the exponents describing anomalous elasticity and transverse undulations, we calculated the tubule diameter, size of the undulations and the anomalous elasticity within the Flory and $\epsilon = d_{uc} - d-$ expansion theories. We have also studied self-avoidance within a Gaussian variational approximation, which unfortunately but, we believe incorrectly predicts that self-avoiding interaction destroys the tubule phase (as it does the crumpled phase) for $d < 4$. We studied the crumpled-to-tubule transition in mean-field theory and with the

$\epsilon = 4 - D$-expansion. Finally we developed a scaling theory of the crumpled-to-tubule and tubule-to-flat transitions.

Our *exact* predictions for the phantom tubules Eqs.5.15,5.16,5.38,5.48 have been *quantitatively* verified in the recent simulations by the authors of Ref. 2.

The possibility of the existence of a new tubule phase intermediate between the fully disordered crumpled phase and fully ordered flat phase is exciting from both basic physics and potential applications points of view. Recently much attention has focussed on utilizing self-assembled microstructures for encapsulations for various applications, most notably controlled and slow drug delivery.[10] The structural stability of polymerized membranes is superior to their liquid membrane analogs. The theoretical discovery of the tubule phase significantly expands the number of possibilities, and also offers the potential tunability (by, e.g., adjusting the strength of self-avoidance) of the tubule diameter and therefore the amount of encapsulation and rate of delivery.

The realization of the tubule phase in polymerized membranes carries even more significance if the claims that the fully crumpled phase in polymerized membranes does not exist are in fact correct, since in this case the tubule phase is the only disordered phase of a polymerized membrane.

With the recent focus on self-assembly, it may be possible in the near future to freeze in intrinsic anisotropy by polymerizing tilted phase of liquid membranes or cross-linking polymers. Further numerical simulations which include self-avoidance offer another avenue to investigate our predictions. We hope that our work stimulates further theory, simulations and experiments in this area.

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